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Thermodynamics and Fluid Mechanics 2

Fluids Topic 2: Boundary layer flows

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Topic 2 – Boundary layer flows

Topic 2 can be studied in F. White, Ch. 6/7

- Definitions
- Boundary layer thickness
- Displacement thickness
- Momentum thickness
- Integral analysis
- Laminar boundary layers: von Karman solution
- Laminar boundary layers: Blasius solution
- Turbulent boundary layers
- Law of the wall
- Effect of roughness
- Effect of pressure gradient

Topic 2 – Boundary layer flows

Learning outcomes:

- Be able to describe how a boundary layer develops on a flat plate
- Know what a boundary layer velocity profile is and be capable of sketching both for laminar and turbulent flow
- Know what methods are used to define boundary layer thickness and be able to explain the difference between δ, δ^* and θ
- Understand the relationships between plate drag, momentum thickness and shear stress
- Understand how shear stress and drag are related for a flat plate
- Be able to calculate boundary layer thickness, drag force and local shear stress for a flat plate with both laminar and turbulent boundary layer
- Understand how flow condition varies through a turbulent boundary layer
- Be able to explain what the “law of the wall” chart shows
- Understand how roughness affects drag and perform appropriate calculations
- Be capable of describing how pressure gradient affects boundary layer profile

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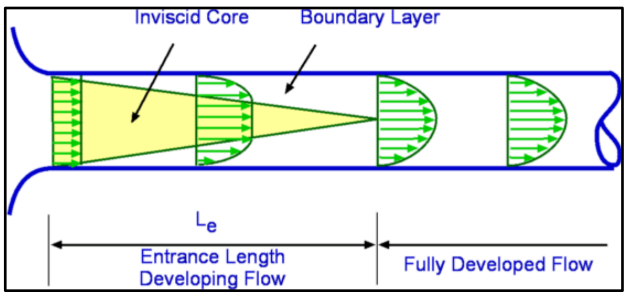
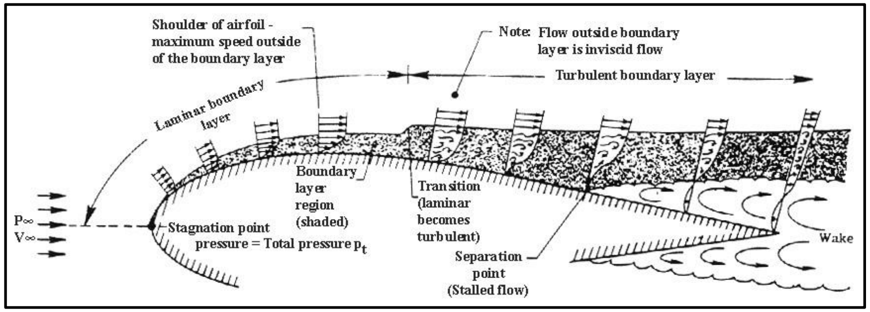
Boundary layers

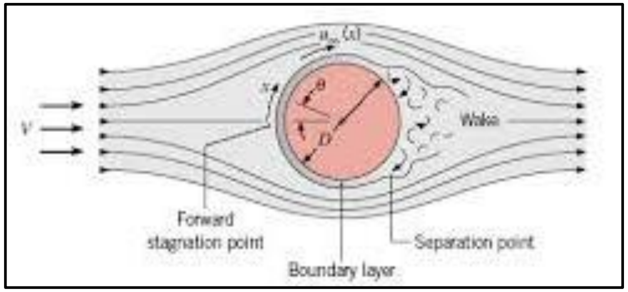
When a fluid flows in contact with a solid surface, the no-slip condition at the wall generates a flow region near the wall where viscous effects are important, and the velocity increases rapidly from zero (at the wall) to its undisturbed main stream value (far from the wall). This region is the **boundary layer**, and it exists in both internal and external flows. Outside the boundary layer, viscous effects are negligible, and the flow can be treated as inviscid.

Boundary layer in pipe flow



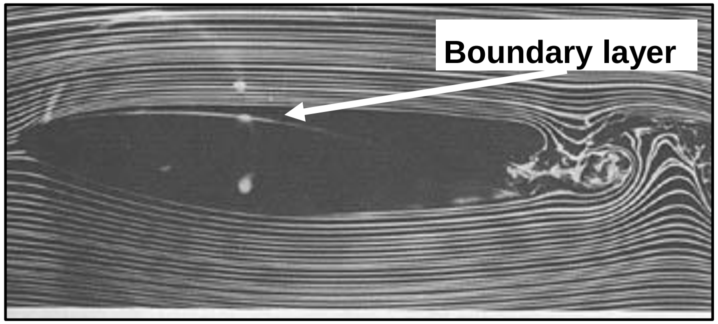
Boundary layer over an airfoil





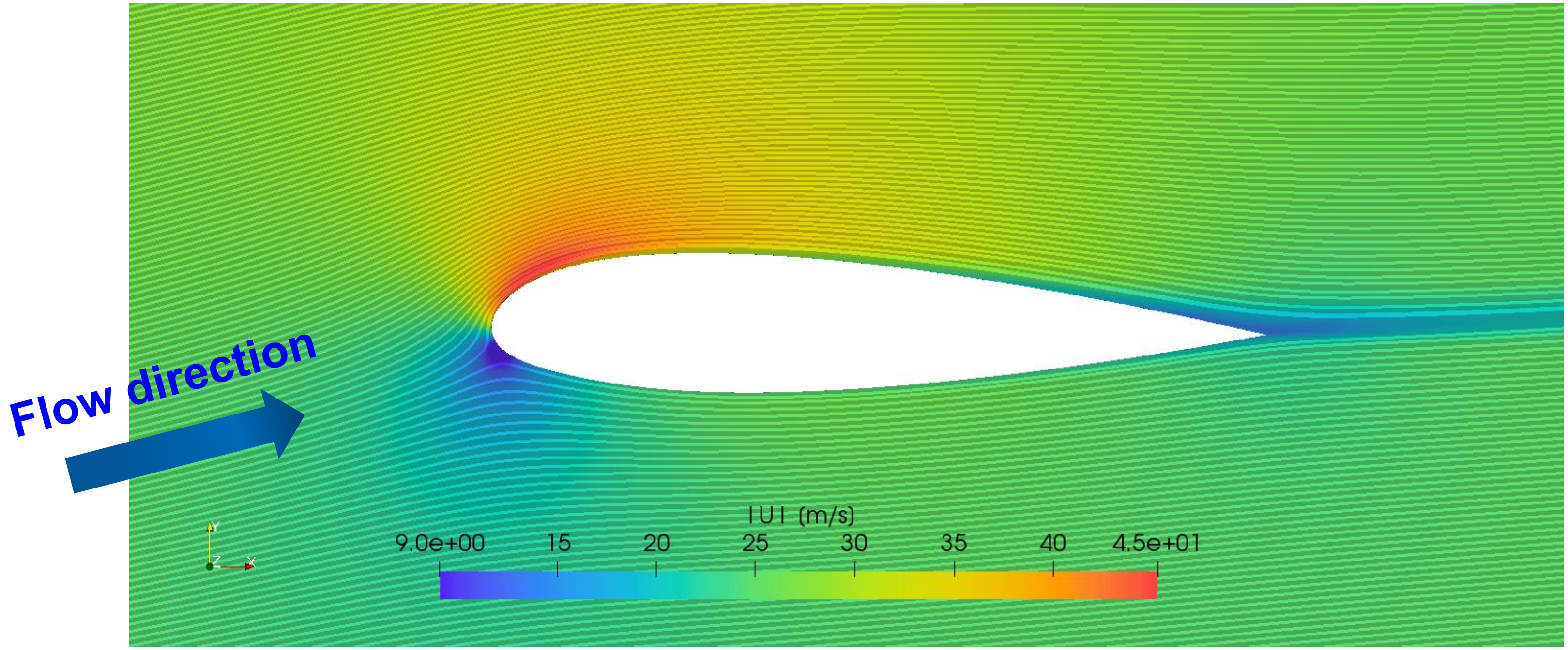
Boundary layer around a cylinder

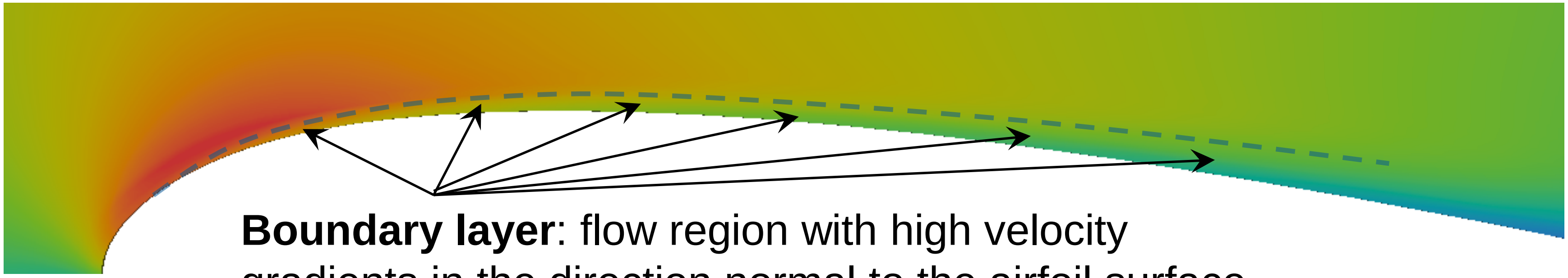
Boundary layer over an airfoil (experiment)



Boundary layers

Flow past an airfoil: CFD simulation





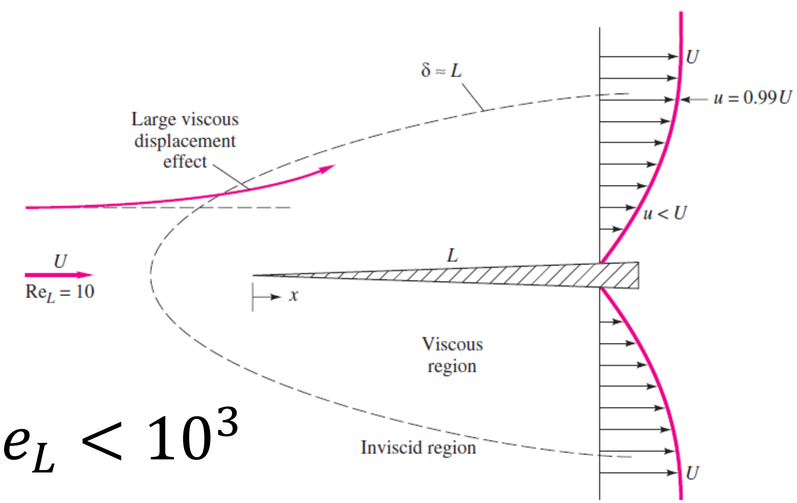
Boundary layer: flow region with high velocity gradients in the direction normal to the airfoil surface

Boundary layers

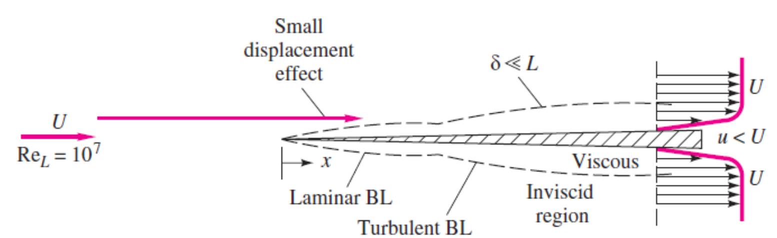
The dynamics of the boundary layer depends on the **Reynolds number**. For flow over a flat plate: $Re_L = \frac{\rho U L}{\mu}$, $Re_x = \frac{\rho U x}{\mu}$ x: distance from leading edge

- Low-Re: $Re_L < 10^3 \Longrightarrow$ Broad viscous region, $\delta \approx L$, extending ahead of the plate
- High-Re: $Re_L > 10^3 \Longrightarrow$ Thin boundary layer, $\delta \ll L$, effects only downstream
  - $10^3 < Re_L < 10^6 \Longrightarrow$ Laminar boundary layer
  - $10^6 < Re_L \Longrightarrow$ Turbulent boundary layer

They can occur in sequence: laminar till $Re_x = \frac{\rho U x}{\mu} = 10^6$, turbulent for $Re_x > 10^6$



$Re_L < 10^3$

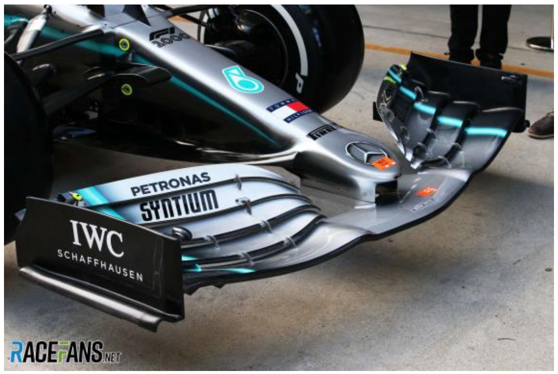


$Re_L > 10^3$

Boundary layers

In practical engineering applications Reynolds numbers are high.

Example: flow past the front wing of a F1 car; $U = 360\ km/h, L = 0.4\,m$

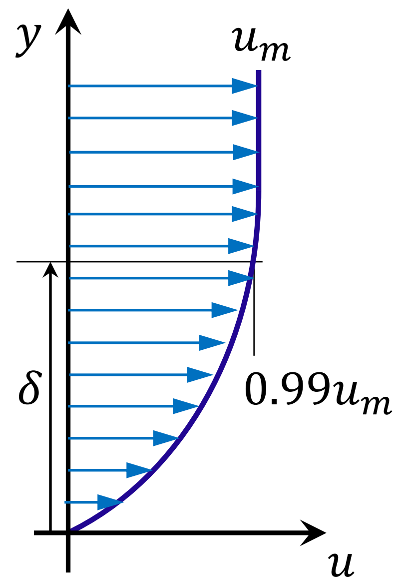


$$Re_L = \frac{\rho U L}{\mu} = \frac{1.2\,\frac{kg}{m^3} \cdot 100\,\frac{m}{s} \cdot 0.4\,m}{1.82 \cdot 10^{-5} Pa \cdot s} = 2.6 \cdot 10^6$$

Example: flow past the wing of an airplane; $U = 800\ km/h, L = 2\,m$, air properties @10km altitude

$$Re_L = \frac{\rho U L}{\mu} = \frac{0.4\,\frac{kg}{m^3} \cdot 222\,\frac{m}{s} \cdot 2\,m}{1.45 \cdot 10^{-5} Pa \cdot s} = 1.22 \cdot 10^7$$

Boundary layers thickness



δ: distance from the surface at which the velocity reaches 99% of the velocity of the main stream.

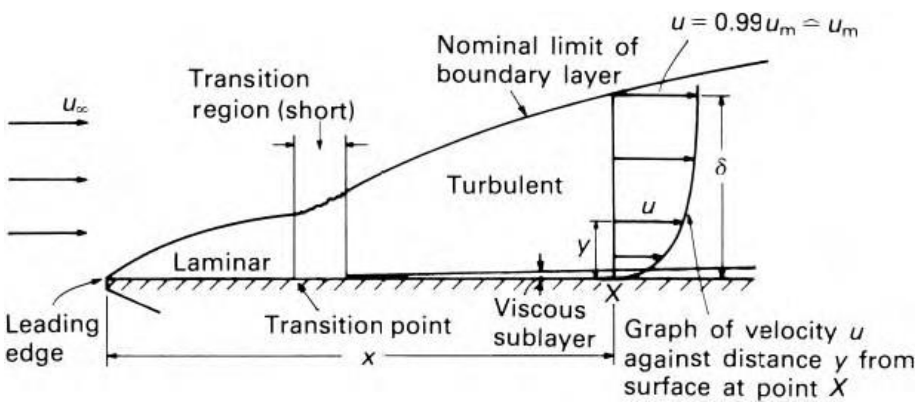
It increases with x (will see later):

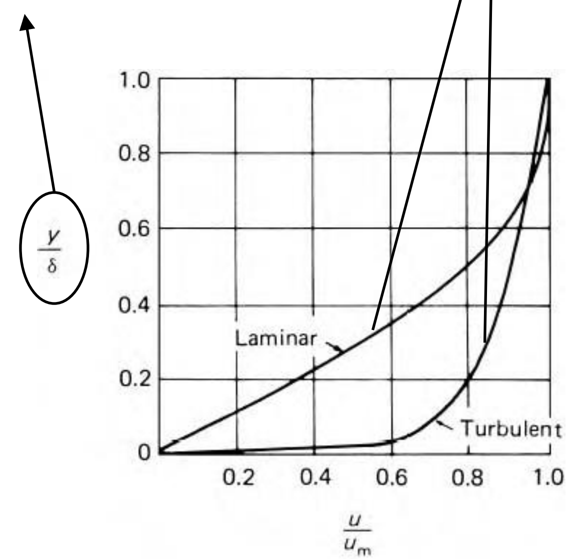
- $\delta \approx 5x/Re_x^{1/2}, \quad 10^3 < Re_x < 10^6$
- $\delta \approx 0.16x/Re_x^{1/7}, \quad Re_x > 10^6$

Velocity gradient at the wall is much larger for turbulent flow

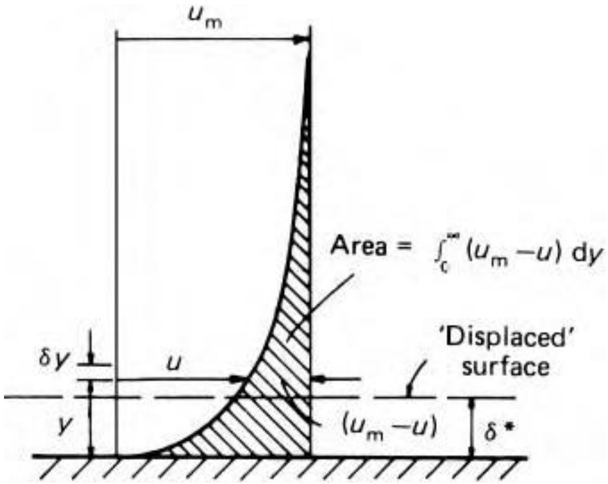
Distance from surface rescaled by boundary layer thickness

δ increases more rapidly when flow becomes turbulent





Displacement thickness



Other measures for thickness:

displacement thickness δ^*

The wall slows down the fluid and thus the volumetric flow rate of fluid is reduced. δ^* quantifies the vertical displacement of the wall that would have caused the same flow rate reduction if $u(x,y) = u_m$ everywhere.

Actual volumetric flow rate above the plate: $\int_0^\infty u(x,y)dy$

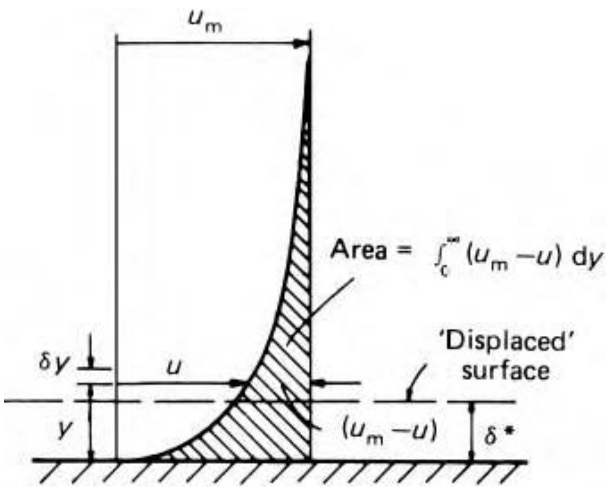
Flow rate above the plate if there was no boundary layer : $\int_0^\infty u_m dy$

Volumetric flow rate reduction: $\int_0^\infty (u_m - u)dy$

Flow rate within distance δ^* if there was no boundary layer : $\int_0^{\delta^*} u_m dy = u_m \delta^*$

$$u_m \delta^* = \int_0^\infty (u_m - u)dy \Rightarrow \boxed{\delta^* = \int_0^\infty \left(1 - \frac{u}{u_m}\right) dy}$$

Momentum thickness



Other measures for thickness:

momentum thickness θ

vertical displacement of the wall that, with the same mass flow rate ρu, would have caused the same momentum reduction if $u(x,y) = u_m$ everywhere.

Careful: same mass flow rate ρu

Actual momentum flux above the plate: $\int_0^\infty (\rho u) u\,dy$

Momentum flux if there was no boundary layer: $\int_0^\infty (\rho u) u_m\,dy$

→ Momentum reduction: $\int_0^\infty \rho u (u_m - u)\,dy$

Momentum flux within distance θ if there was no boundary layer: $(\rho u_m) u_m \theta$

$$(\rho u_m) u_m \theta = \int_0^\infty \rho u (u_m - u)\,dy \Rightarrow \boxed{\theta = \int_0^\infty \frac{u}{u_m}\left(1 - \frac{u}{u_m}\right) dy}$$

Integral analysis of the boundary layer

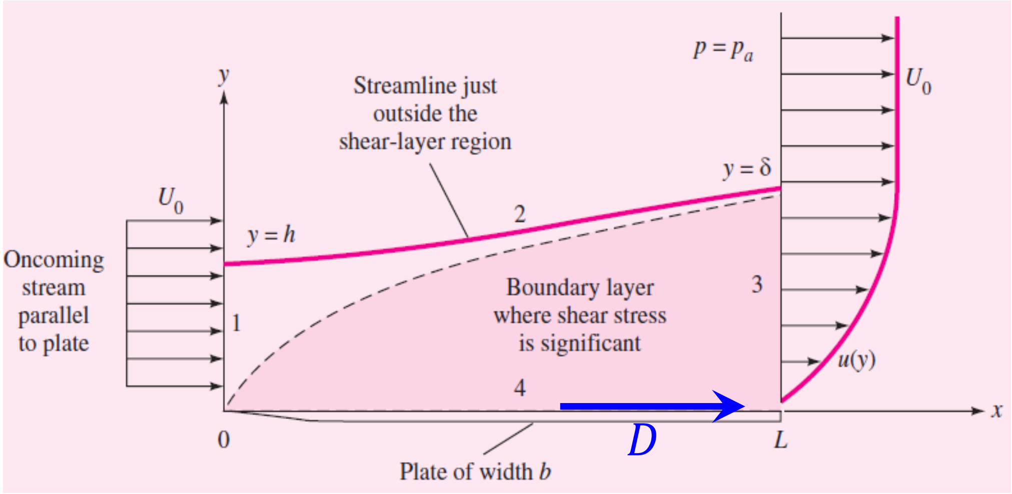
A practical question: what's the drag force D exerted by the fluid to the plate?

This can be obtained by integrating the momentum equation on a control volume placed above the flat plate.

We consider the control volume with boundary surface 1234 in the figure below.

At steady-state, the notes of TF1 (linear momentum, p.100) and the slides of T1-Navier-Stokes (slide 15) suggest: $F_x = \dot{Q}_{x,out} - \dot{Q}_{x,in}$

Flow is null across surfaces 2 (it's a streamline: fluid doesn't cross it) and 4 (it's a wall)



$$\dot{Q}_{x,in} = \dot{Q}_{x,1} = (\rho b h U_0) U_0$$

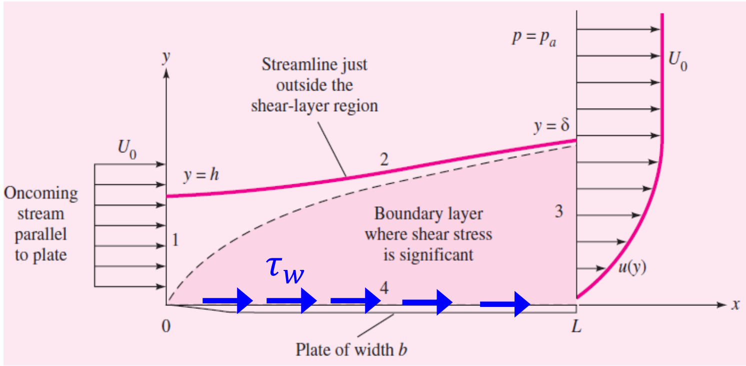
$$\dot{Q}_{x,out} = \dot{Q}_{x,3} = \int_0^{\delta} [\rho b u(L,y)] u(L,y) dy$$

$F_x = -D,$

- ➢ We assume constant pressure everywhere and therefore neglect the force induced by pressure gradients, we retain only drag.
- ➢ F_x is negative because it's the force exerted by the plate to the fluid and thus directed towards $-x$.

Integral analysis of the boundary layer

$$-D = \rho b \int_0^\delta u^2 dy - \rho b h U_0^2$$



Let's now find an expression for h.

Continuity: $\dot{M}_{out} - \dot{M}_{in} = 0$

$$\Rightarrow \int_0^\delta \rho b u dy - \rho b h U_0 = 0 \Rightarrow h = \int_0^\delta \frac{u}{U_0} dy$$

and use it in the expression for D:

$$D = \rho b \int_0^\delta u(U_0 - u) dy = \boxed{\rho b U_0^2 \theta}$$

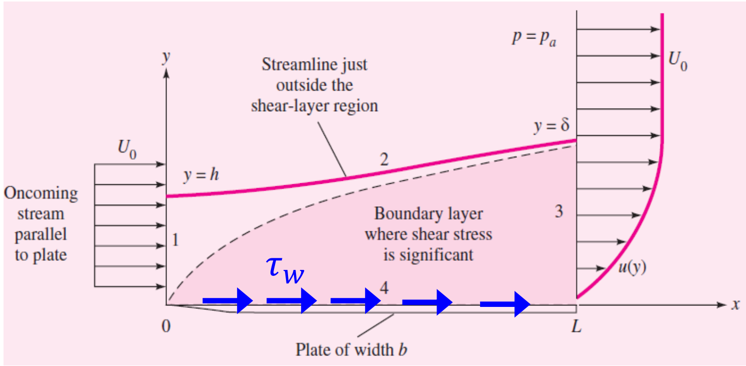
Momentum thickness and drag are related!

Having neglected pressure gradients, the drag force is only due to the wall shear stress and the total drag over the plate is the integral of the wall shear stress:

$$D = b \int_0^L \tau_w(x) dx \Rightarrow \frac{dD}{dx} = b\tau_w(x) = \rho b U_0^2 \frac{d\theta}{dx} \Rightarrow \boxed{\tau_w(x) = \rho U_0^2 \frac{d\theta}{dx}}$$

Integral analysis of the boundary layer

$$\tau_w(x) = \rho U_0^2 \frac{d\theta}{dx}$$



This expression is valid for both laminar and turbulent flows. However, we cannot use it unless we know how to express $d\theta/dx$. For this, we'll need a velocity profile.

The shear stress is often expressed with the non-dimensional factor called skin friction coefficient (see Topic 1, slide 42):

$$C_f(x) = \frac{\tau_w(x)}{\frac{1}{2}\rho U_0^2} = 2\frac{d\theta}{dx}$$

The drag can also be expressed in terms of a non-dimensional drag coefficient:

$$C_D = \frac{D/plate\_area}{\frac{1}{2}\rho U_0^2} = \frac{D/(bL)}{\frac{1}{2}\rho U_0^2}$$

Worked example 1

A thin, flat plate of $2\,m$ length is placed parallel to a $6\,m/s$ flow of water at $20\,°C$.

- Calculate the boundary layer thickness at the trailing edge using the relationships of slide 8.
- Calculate the length of the plate where the flow is laminar.

Solution

Water at $20\,°C$: $\rho = 998\ kg/m^3, \mu = 0.001\ Pa \cdot s$.

$$Re_L = \frac{\rho U L}{\mu} = \frac{998\,\frac{kg}{m^3} \cdot 6\,\frac{m}{s} \cdot 2\,m}{0.001\ Pa \cdot s} = 1.2 \cdot 10^7 > 10^6 \longrightarrow$$

The flow becomes <u>turbulent</u>

Boundary layer thickness for turbulent flow (slide 8):

$$\delta \approx \frac{0.16x}{Re_L^{1/7}} = \frac{0.16 \cdot 2\,m}{(1.2 \cdot 10^7)^{1/7}} = 0.027\,m$$

Plate length in laminar flow: $Re_{lam} = \frac{\rho U L_{lam}}{\mu} = 10^6 \Rightarrow L_{lam} = \frac{10^6 \mu}{\rho U} = 0.167\,m$

Worked example 2

A long, thin, flat plate is placed parallel to a $100\ m/s$ flow of air at $20\ ℃$ and $p = 1\ bar$.

- Calculate at what distance from the leading edge the boundary layer thickness will be $2\ cm$.

Solution

Air at $20\ ℃, 1\ bar$: $\rho = 1.2\ \ kg/m^3, \mu = 1.8 \cdot 10^{-5} Pa \cdot s$.

If the flow was laminar, the necessary length would be:

$$\left.\frac{\delta}{x}\right|_{lam} = \frac{5}{\sqrt{Re_x}} = 5\sqrt{\frac{\mu}{\rho U x}} \Longrightarrow x = \frac{\rho U \delta^2}{5^2 \mu} = \frac{1.2\frac{kg}{m^3} * 100\frac{m}{s} * (0.02\ m)^2}{25 * 1.8 * 10^{-5} Pa * s} = 106\ m$$

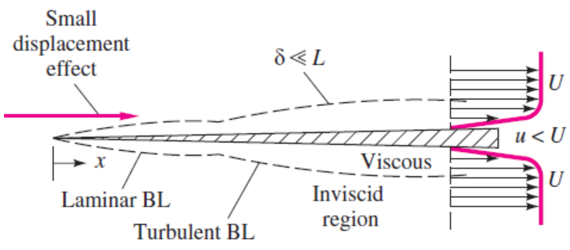
To which it corresponds: $Re_x = \frac{\rho U x}{\mu} = 7 \cdot 10^8$ Therefore the flow is turbulent and cannot use the equation above!

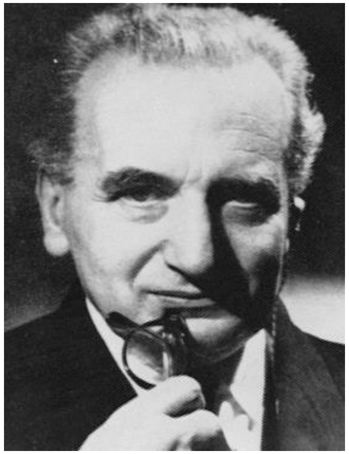
Let's try with: $\left.\frac{\delta}{x}\right|_{turb} = \frac{0.16}{{Re_x}^{1/7}} \Longrightarrow x = \frac{\delta^{7/6}}{0.16^{7/6}}\left(\frac{\rho U}{\mu}\right)^{1/6} = 1.21\ \mathrm{m}$

To which it corresponds: $Re_x = \frac{\rho U x}{\mu} = 7.9 \cdot 10^6$ The use of the relationship for turbulent flow is justified

Laminar boundary layers

The analysis developed so far is valid for both laminar and turbulent boundary layers. We have seen how to express τ_w, but we don't know yet how to calculate it based on known parameters (ρ, U_0, μ, x). In order to get a numerical result, we need to assume a velocity profile.





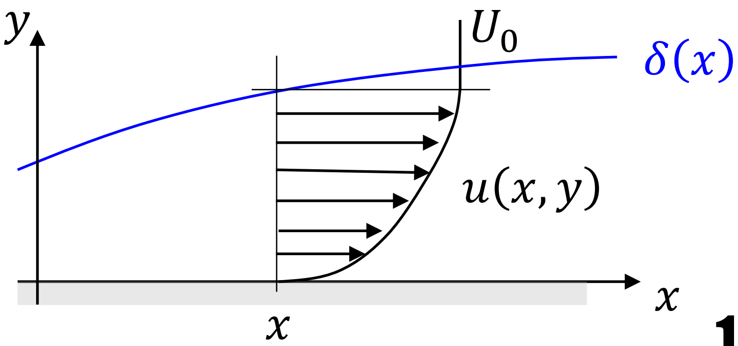
Laminar flow – von Karman approximate solution

For a laminar flow, von Karman (1921) assumed the following parabolic velocity profile:

$$u(x,y) = U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right), \qquad 0 \le y \le \delta(x)$$

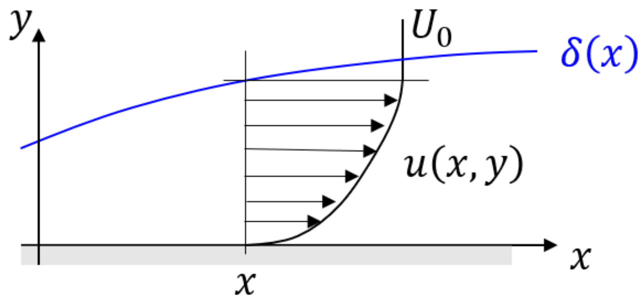
constructed to satisfy the boundary conditions:

- $u(y=\delta) = U_0$
- $\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0$
- $u(y=0) = 0$
- $\left.\frac{\partial u}{\partial y}\right|_{y=0} \neq 0$



Laminar boundary layers – von Karman

$$u(x,y) = U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right), \qquad 0 \le y \le \delta(x)$$



We can now use this velocity profile to derive all the quantities of interest.

Because for $y > \delta, u \approx U_0$

$$\theta(x) = \int_0^\infty \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy = \int_0^\delta \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right) dy = \frac{2}{15}\delta(x)$$

$$\rightarrow \tau_w(x) = \rho U_0^2 \frac{d\theta}{dx} = \rho U_0^2 \frac{2}{15}\frac{d\delta}{dx}$$

Not yet useful, now we need $\frac{d\delta}{dx}$

But remember that: $\tau_w(x) = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{2\mu U_0}{\delta(x)}$

$$\rightarrow \rho U_0^2 \frac{2}{15}\frac{d\delta}{dx} = \frac{2\mu U_0}{\delta(x)}$$

rearranging →

$$\delta\, d\delta = \frac{15\mu}{\rho U_0} dx$$

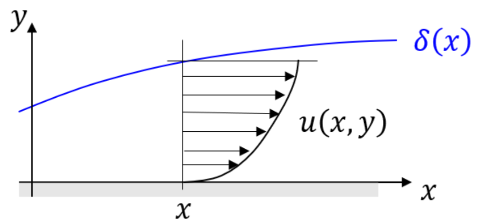
Integrating both sides between $x = 0$ and x →

$$\rightarrow \int_{\delta(x=0)=0}^{\delta(x)} \delta\, d\delta = \int_{x=0}^{x} \frac{15\mu}{\rho U_0} dx \rightarrow \left[\frac{\delta^2}{2}\right]_0^\delta = \frac{\delta^2}{2} = \frac{15\mu}{\rho U_0} x \rightarrow \frac{\delta}{x} \approx 5.5\sqrt{\frac{\mu}{\rho U_0 x}} = \boxed{\frac{5.5}{\sqrt{Re_x}}}$$

Laminar boundary layers – von Karman

$$\delta \approx \frac{5.5x}{\sqrt{Re_x}}$$

In laminar flow, the boundary layer thickness grows as $x^{1/2}$



$$\delta^*(x) = \int_0^\delta \left(1 - \frac{u}{U_0}\right) dy = \int_0^\delta \left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right) dy = \frac{\delta(x)}{3} = \frac{1.83x}{\sqrt{Re_x}}$$

$$\theta(x) = \frac{2}{15}\delta(x) = \frac{0.73x}{\sqrt{Re_x}}$$

$$\tau_w(x) = \frac{2\mu U_0}{\delta(x)} = \frac{0.363\rho U_0^2}{\sqrt{Re_x}}$$

Results valid as long as: $10^3 < Re_x < 10^6$

$$C_f(x) = \frac{\tau_w(x)}{\frac{1}{2}\rho U_0^2} = \frac{0.73}{\sqrt{Re_x}}$$

$$D = b\int_0^L \tau_w(x) dx = 0.73\frac{\rho U_0^2 bL}{\sqrt{Re_L}} \quad \Rightarrow \quad C_D = \frac{D/(bL)}{\frac{1}{2}\rho U_0^2} = \frac{1.46}{\sqrt{Re_L}}$$

Is this the only theory available? No, there is also Blasius' "exact" solution!

Laminar boundary layers – Blasius

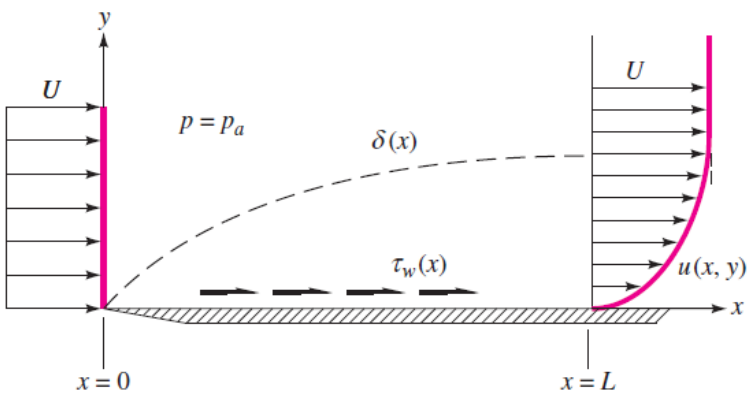
Prandtl's boundary layer theory

The flow over the plate is assumed steady-state, incompressible, the fluid Newtonian; gravity is neglected. The equations governing the flow are (T1, slide 30):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

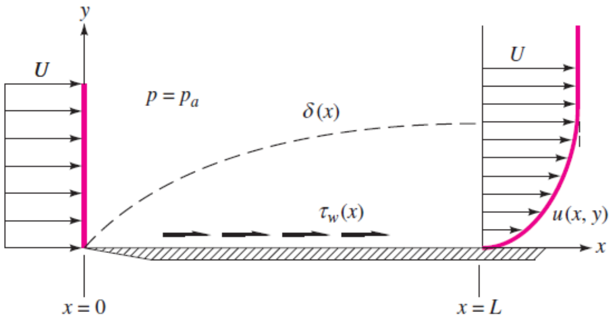


Prandtl applied these equations to the flow in the boundary layer (1904), under the assumption that $\delta \ll x$, and he simplified them by assuming that:

- $v \ll u$: the velocity is almost uni-directional along x
- $\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x} \ll \frac{\partial v}{\partial y}$: gradients along y are more important than those along x
- $Re_x \gg 1$

Laminar boundary layers – Blasius

Therefore, the equations could be simplified as:



$$\rho\left[u\cancel{\frac{\partial v}{\partial x}}+v\cancel{\frac{\partial v}{\partial y}}\right]=-\frac{\partial p}{\partial y}+\mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}}+\cancel{\frac{\partial^2 v}{\partial y^2}}\right) \longrightarrow p=p(x)$$

small   small   very small   small

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{dp}{dx}+\mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}\right) \longrightarrow u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

small

The continuity equation stays unchanged: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

Blasius (1908) further simplified these equations by assuming that, for a flat plate,

$\frac{dp}{dx}=0$. Furthermore, he argued that:

$$\frac{u(x,y)}{U_0}=f\left(\frac{y}{\delta(x)}\right),$$ meaning that the profile is <u>self-similar</u>.

Based on this theory, he obtained a numerical solution for $u(x,y)$, details here:

https://en.wikipedia.org/wiki/Blasius_boundary_layer

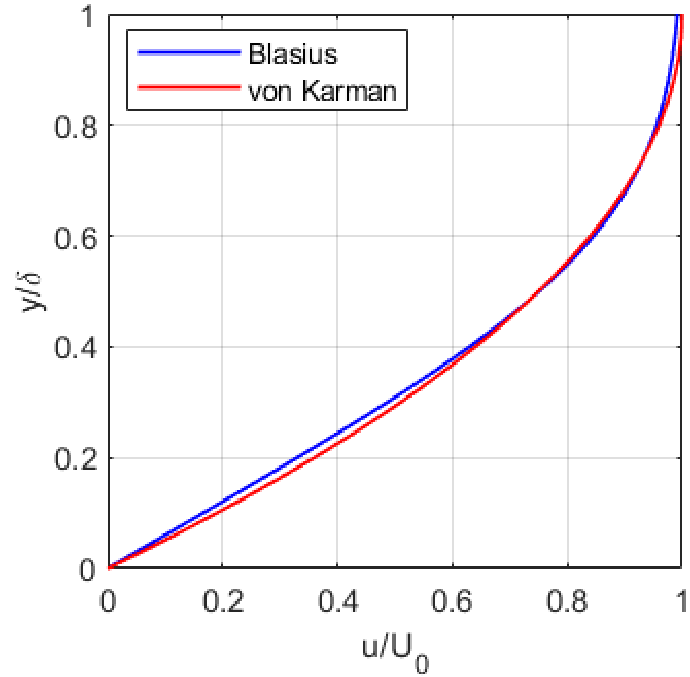
or (Ch. 3): https://essay.utwente.nl/63314/1/BSc_report_Peter_Puttkammer.pdf

Laminar boundary layers – Blasius

Blasius numerical solution:

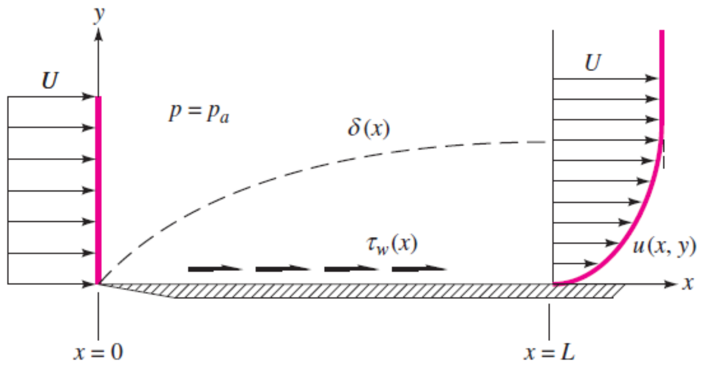
$y[U/(\nu x)]^{1/2}$	u/U	$y[U/(\nu x)]^{1/2}$	u/U
0.0	0.0	2.8	0.81152
0.2	0.06641	3.0	0.84605
0.4	0.13277	3.2	0.87609
0.6	0.19894	3.4	0.90177
0.8	0.26471	3.6	0.92333
1.0	0.32979	3.8	0.94112
1.2	0.39378	4.0	0.95552
1.4	0.45627	4.2	0.96696
1.6	0.51676	4.4	0.97587
1.8	0.57477	4.6	0.98269
2.0	0.62977	4.8	0.98779
2.2	0.68132	5.0	0.99155
2.4	0.72899	∞	1.00000
2.6	0.77246		

Comparison of velocity profiles of Blasius and von Karman



$$\frac{u}{U_0} \approx 0.99, y \equiv \delta$$

$$\delta\sqrt{\frac{\rho U_0}{\mu x}} \approx 5.0 \Longrightarrow \frac{\delta}{x} \approx \frac{5.0}{\sqrt{Re_x}}$$



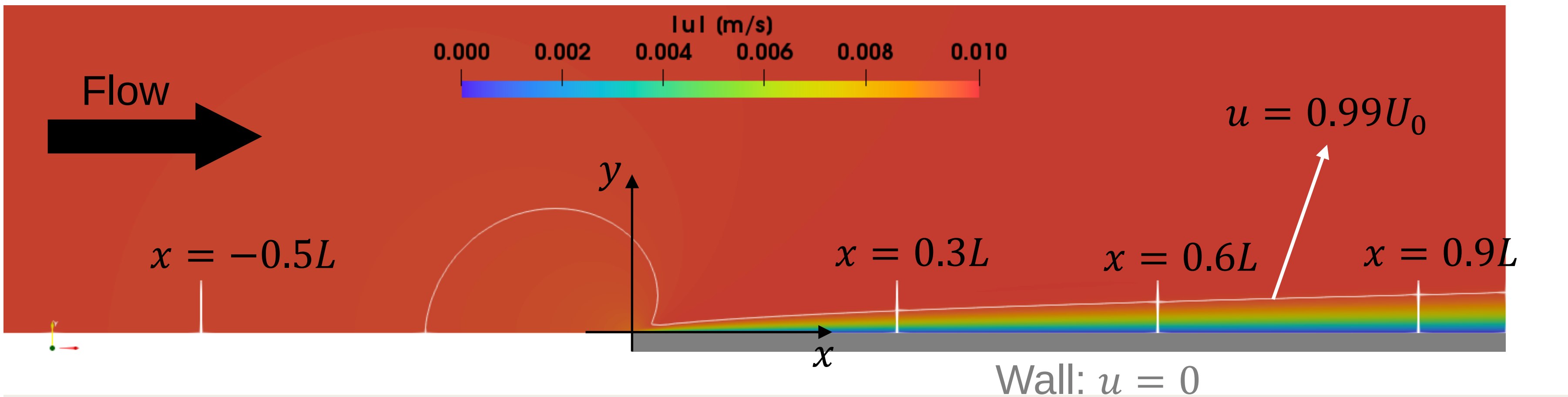
$$\frac{\delta^*}{x} = \frac{1.721}{\sqrt{Re_x}}, \frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}, C_f = \frac{0.664}{\sqrt{Re_x}}, C_D = \frac{1.328}{\sqrt{Re_L}}$$

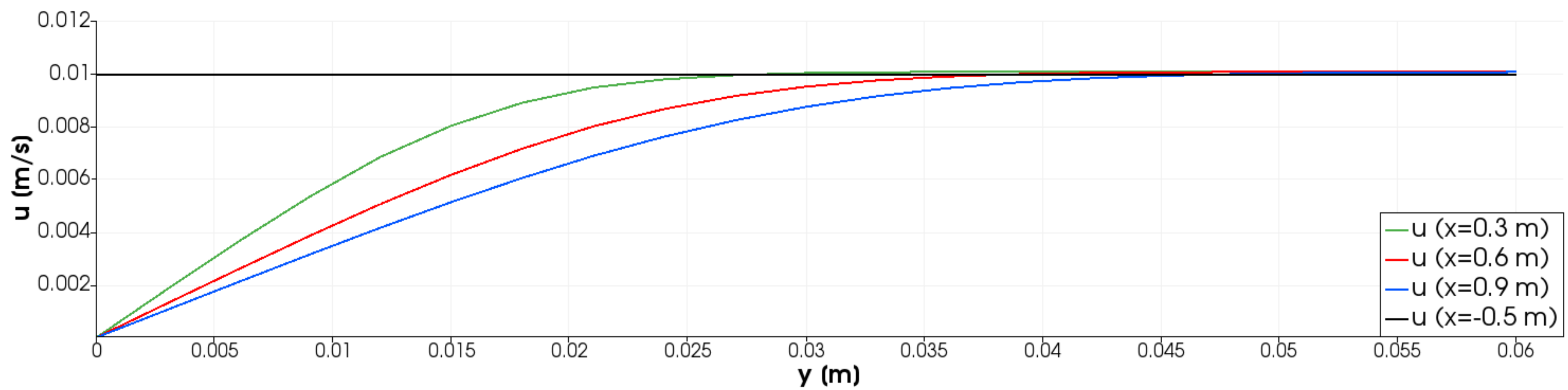
Laminar boundary layers – Blasius

Very nice theories Dr. Blasius and Dr. von Karman, but do they work??

CFD simulation of flow over a flat plate: $U_0 = 0.01\, m/s\,, \nu = \mu/\rho = 10^{-6}\, m^2/s\,, L = 1\, m$.

$Re_L = 10^4$





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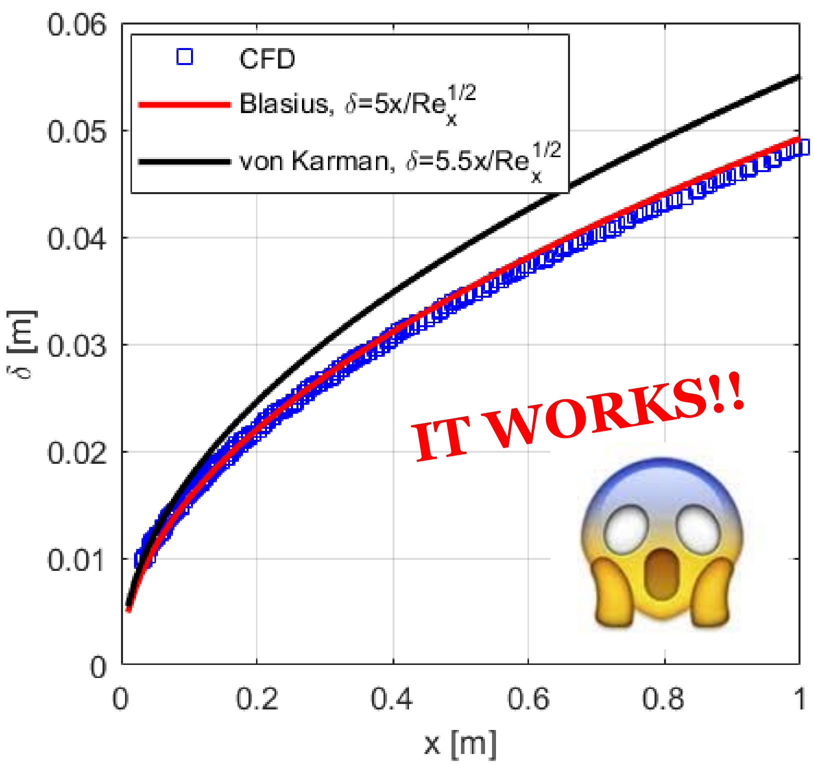
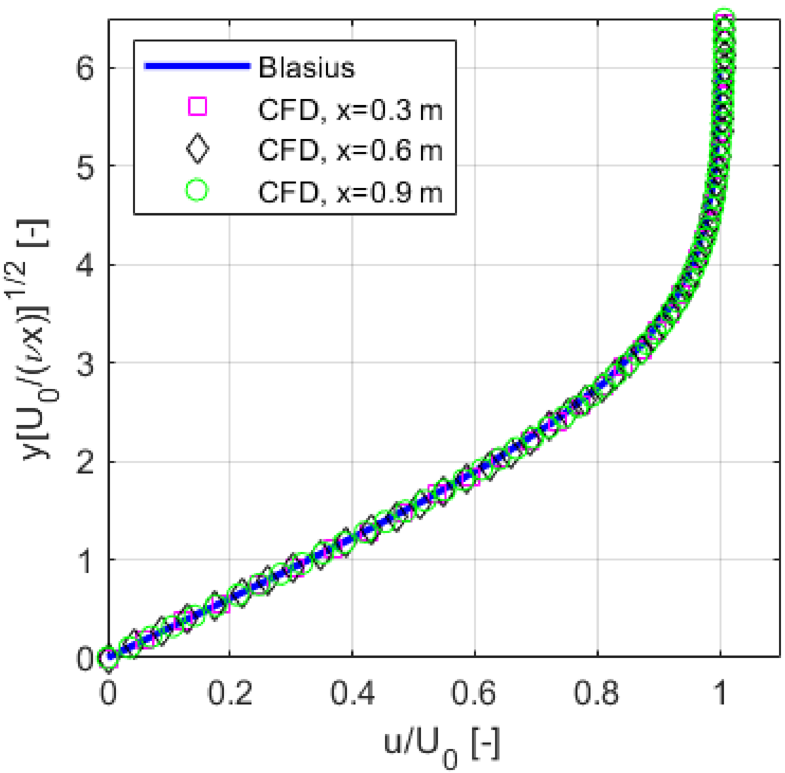




From CFD: $\delta(x)$ is taken as the line where $u = 0.99U_0$, see the white line in the contour plot of past slide

Worked example 3

A laminar flow of air moves parallel to a flat plate of length $L = 0.6\,m$ and width $b = 0.3\,m$, with a velocity $U = 5\ m/s$. Estimate at the trailing edge of the plate, the boundary layer thickness, δ, the displacement thickness, δ^*, and the momentum thickness, θ. In your estimation use Blasius boundary layer exact solution. For air, take $\rho = 1.2\ kg/m^3$ and $\mu = 1.8 \cdot 10^{-5}\ Pa \cdot s$.

Solution

$$Re_L = \frac{\rho U L}{\mu} = \frac{1.2\,\frac{kg}{m^3} \cdot 5\,\frac{m}{s} \cdot 0.6\,m}{0.000018\,Pa \cdot s} = 200000$$

$$\delta(x = L) = \frac{5.0L}{\sqrt{Re_L}} = \frac{5 \cdot 0.6\,m}{\sqrt{200000}} = 6.71\,mm$$

$$\delta^*(x = L) = \frac{1.721L}{\sqrt{Re_L}} = 2.31\,mm$$

$$\theta(x = L) = \frac{0.664L}{\sqrt{Re_L}} = 0.89\,mm$$

Worked example 4

A sharp flat plate with length $L = 50\ cm$ and width $b = 3\ m$ is parallel to a stream of velocity $U = 2.5\ m/s$. Find the drag on one side of the plate, and the boundary layer thickness δ at the trailing edge, for (a) air and (b) water at 20 ℃ and 1 atm

Solution

(a) Air at 20 ℃, 1 atm: $\rho = 1.2\ kg/m^3, \mu = 1.8 \cdot 10^{-5} Pa \cdot s$.

$$Re_L = \frac{\rho U L}{\mu} = \frac{1.2\ \frac{kg}{m^3} \cdot 2.5 \frac{m}{s} \cdot 0.5\ m}{0.000018\ Pa \cdot s} = 83333$$ → Flow is laminar

$$\delta(x = L) = \frac{5.0L}{\sqrt{Re_L}} = \frac{5 \cdot 0.5\ m}{\sqrt{83333}} = 8.66\ mm$$

$$C_D = \frac{1.328}{\sqrt{Re_L}} = 0.0046$$

$$C_D = \frac{D/(bL)}{\frac{1}{2}\rho U_0^2} \Longrightarrow D = \frac{1}{2} C_D \rho U_0^2 b L = 0.026\ N$$

Worked example 4

A sharp flat plate with length $L = 50\ cm$ and width $b = 3\ m$ is parallel to a stream of velocity $U = 2.5\ m/s$. Find the drag on one side of the plate, and the boundary layer thickness δ at the trailing edge, for (a) air and (b) water at $20\ ℃$ and $1\ atm$

Solution

(b) Water at $20\ ℃$, $1\ \mathrm{atm}$: $\rho = 998\ kg/m^3$, $\mu = 0.001\ Pa \cdot s$.

$$Re_L = \frac{\rho U L}{\mu} = \frac{998\ \frac{kg}{m^3} \cdot 2.5 \frac{m}{s} \cdot 0.5\ m}{0.001\ Pa \cdot s} = 1.24 \cdot 10^6$$

→ Flow might be turbulent. Let's assume that the plate is very smooth, so that flow is laminar

$$\delta(x = L) = \frac{5.0L}{\sqrt{Re_L}} = \frac{5 \cdot 0.5\ m}{\sqrt{1.24 \cdot 10^6}} = 2.2\ mm$$

$$C_D = \frac{1.328}{\sqrt{Re_L}} = 0.0012$$

$$C_D = \frac{D/(bL)}{\frac{1}{2}\rho U_0^2} \Longrightarrow D = \frac{1}{2} C_D \rho U_0^2 b L = 5.6\ N$$

Worked example 5

A thin flat plate of size $110\ cm\ \times 55\ cm$ is immersed in a stream of oil ($\rho = 870\ kg/m^3, \mu = 0.104\ Pa \cdot s$) of velocity $6\ m/s$. Compute the friction drag on one side of the plate. Find the drag on one side of the plate if the stream is parallel to the (a) longer side or (b) shorter side.

Solution

(a) $L = 1.1\ m$ and $b = 0.55\ m$.

$$Re_L = \frac{\rho U L}{\mu} = 55211$$ → Flow is laminar

$$C_D = \frac{1.328}{\sqrt{Re_L}} = 0.00565 \quad \Longrightarrow D = \frac{1}{2} C_D \rho U_0^2 b L = 53.6\ N$$

(b) $L = 0.55\ m$ and $b = 1.1\ m$.

$$Re_L = \frac{\rho U L}{\mu} = 27606$$ → Flow is laminar

$$C_D = \frac{1.328}{\sqrt{Re_L}} = 0.008 \quad \Longrightarrow D = \frac{1}{2} C_D \rho U_0^2 b L = 75.7\ N$$

Worked example 5

Recalculate the drag force for the following two configurations:

(a)

(b)

Solution

(a) $L = 4.4\ m$ and $b = 0.55\ m$. $Re_L = \frac{\rho U L}{\mu} = 220844$ → Flow is laminar

$$C_D = \frac{1.328}{\sqrt{Re_L}} = 0.0028 \quad \Longrightarrow D = \frac{1}{2} C_D \rho U_0^2 b L = 106.11\ N$$

(b) $L = 2.2\ m$ and $b = 1.1\ m$. $Re_L = \frac{\rho U L}{\mu} = 110422$ → Flow is laminar

$$C_D = \frac{1.328}{\sqrt{Re_L}} = 0.004 \quad \Longrightarrow D = \frac{1}{2} C_D \rho U_0^2 b L = 151.4\ N$$

Worked example 6

The velocity field of an air laminar boundary layer flow at two points above a flat plate, $y = 2\ mm$ and $y = 3\ mm$, both of them located at a distance x from the leading edge of the plate, are known to be $u(x, y = 2\ mm) = 3\ m/s$ and $u(x, y = 3\ mm) = 4\ m/s$. Using von Karman approximation find; (1) the value of the stream velocity above the boundary layer, (2) the thickness of the boundary layer at the point x and (3) the corresponding value of the skin friction coefficient at x. For air, take $\rho = 1.2\ kg/m^3$ and $\mu = 1.8 \cdot 10^{-5}\ Pa \cdot s$.

Solution

Point 1, $y_1 = 2\ mm$: $u_1 = U_0\left(\frac{2y_1}{\delta} - \frac{{y_1}^2}{\delta^2}\right) = 3\ m/s$

Point 2, $y_2 = 3\ mm$: $u_2 = U_0\left(\frac{2y_2}{\delta} - \frac{{y_2}^2}{\delta^2}\right) = 4\ m/s$

$$\frac{u_1}{u_2} = \frac{\frac{2y_1}{\delta} - \frac{{y_1}^2}{\delta^2}}{\frac{2y_2}{\delta} - \frac{{y_2}^2}{\delta^2}} = \frac{2y_1\delta - {y_1}^2}{2y_2\delta - {y_2}^2} \Longrightarrow \delta = \frac{{y_2}^2\frac{u_1}{u_2} - {y_1}^2}{2y_2\frac{u_1}{u_2} - 2y_1} = 5.5\ mm$$

Worked example 6

$$u_1 = U_0\left(\frac{2y_1}{\delta} - \frac{{y_1}^2}{\delta^2}\right) \Longrightarrow U_0 = \frac{u_1}{\frac{2y_1}{\delta} - \frac{{y_1}^2}{\delta^2}} = 5.04\ m/s$$

$$C_f(x) = \frac{0.73}{\sqrt{Re_x}}, \qquad Re_x = \frac{\rho U_0 x}{\mu}$$ but we don't know x

von Karman: $$\delta = \frac{5.5x}{\sqrt{Re_x}} = 5.5\sqrt{\frac{\mu x}{\rho U_0}} \Longrightarrow x = \frac{\rho U_0 \delta^2}{5.5^2 \mu} = 0.339\ m$$

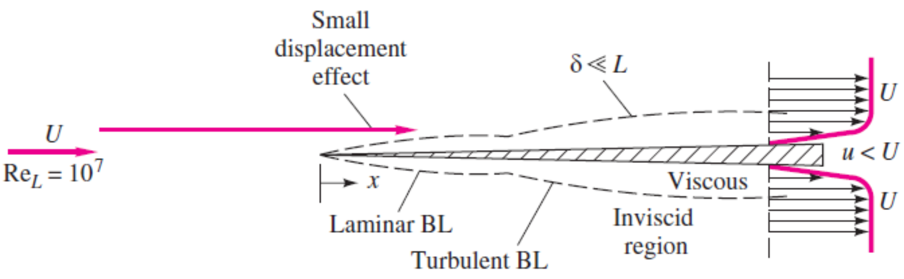
$$Re_x = \frac{\rho U_0 x}{\mu} = 111390$$

$$C_f(x = 0.339\ m) = \frac{0.73}{\sqrt{Re_x}} = 0.00216$$

Turbulent boundary layers

From slide 6: the flow in the boundary layer becomes turbulent when

$$Re_x = \frac{\rho U x}{\mu} > 10^6$$

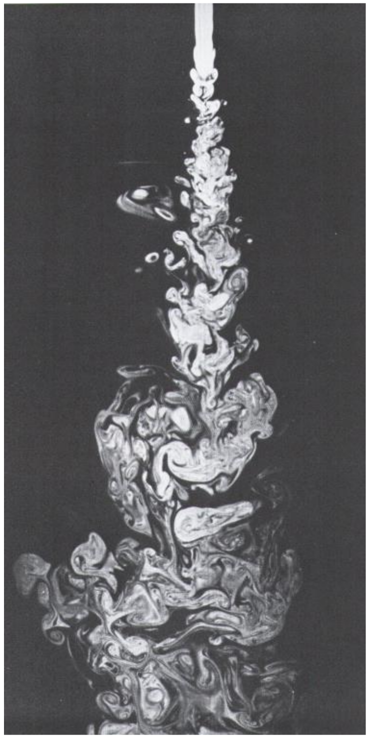


But what is **turbulence**?? Difficult to find a clear definition!

Representation of turbulence by L. da Vinci (1452-1519): "*Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to the random and reverse motion.*"

The starry night (1889) – Van Gogh

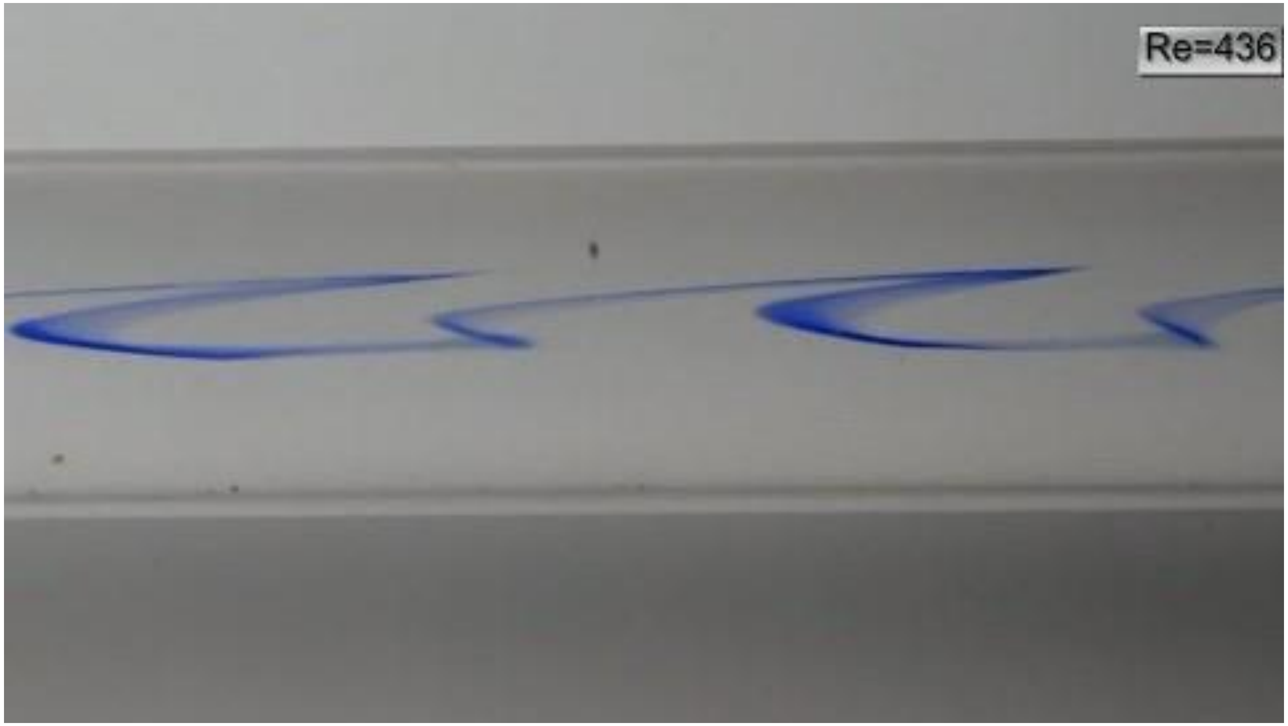
Turbulent water jet



Turbulent boundary layers

Transition from laminar to turbulent flow in a pipe:

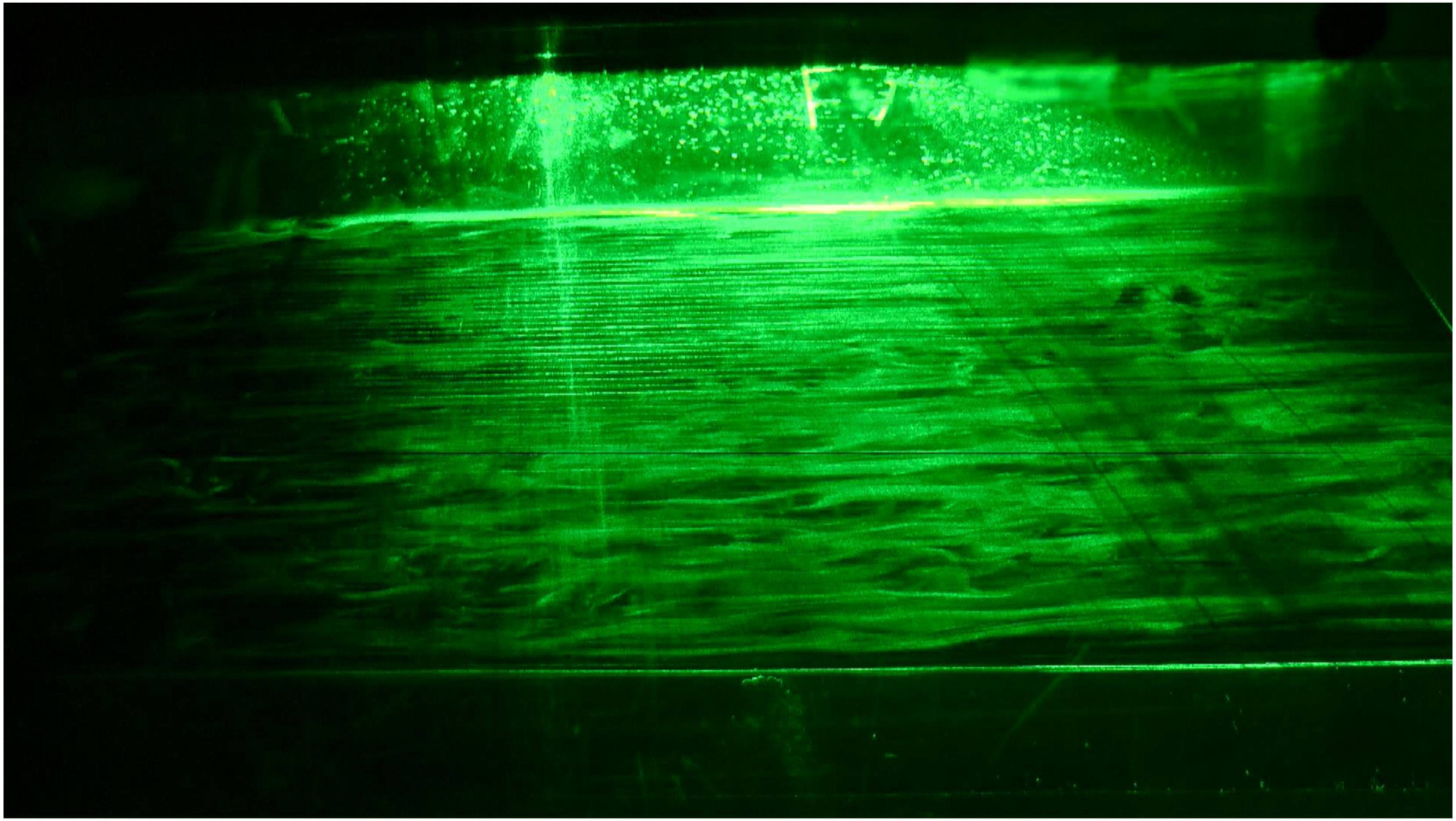
https://www.youtube.com/watch?v=BBiR6FWmyv4



Remember that in a pipe flow becomes turbulent when: $Re = \frac{\rho U D}{\mu} > 2300$

Turbulent boundary layers

Turbulent flow over a flat plate



Turbulence

F. White, Ch. 6.5

Turbulent fluid motion is an irregular flow condition in which velocity and all other flow properties show random and chaotic variations in space and time.

Reynolds decomposition: $u = \bar{u} + u'$

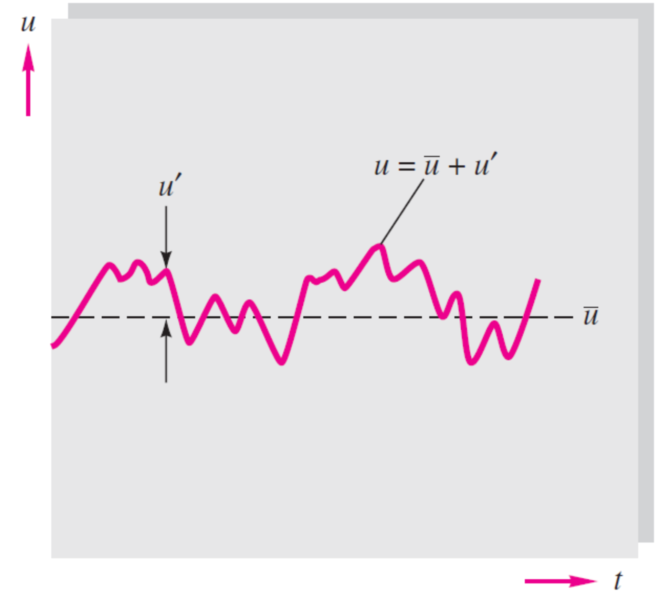
Mean component: $\bar{u} = \frac{1}{T}\int_0^T u\,dt$

T: generic time period

Fluctuation: $u' = u - \bar{u}$

$$\overline{u'} = \frac{1}{T}\int_0^T (u - \bar{u})\,dt = \bar{u} - \bar{u} = 0$$

The fluctuation has zero mean value

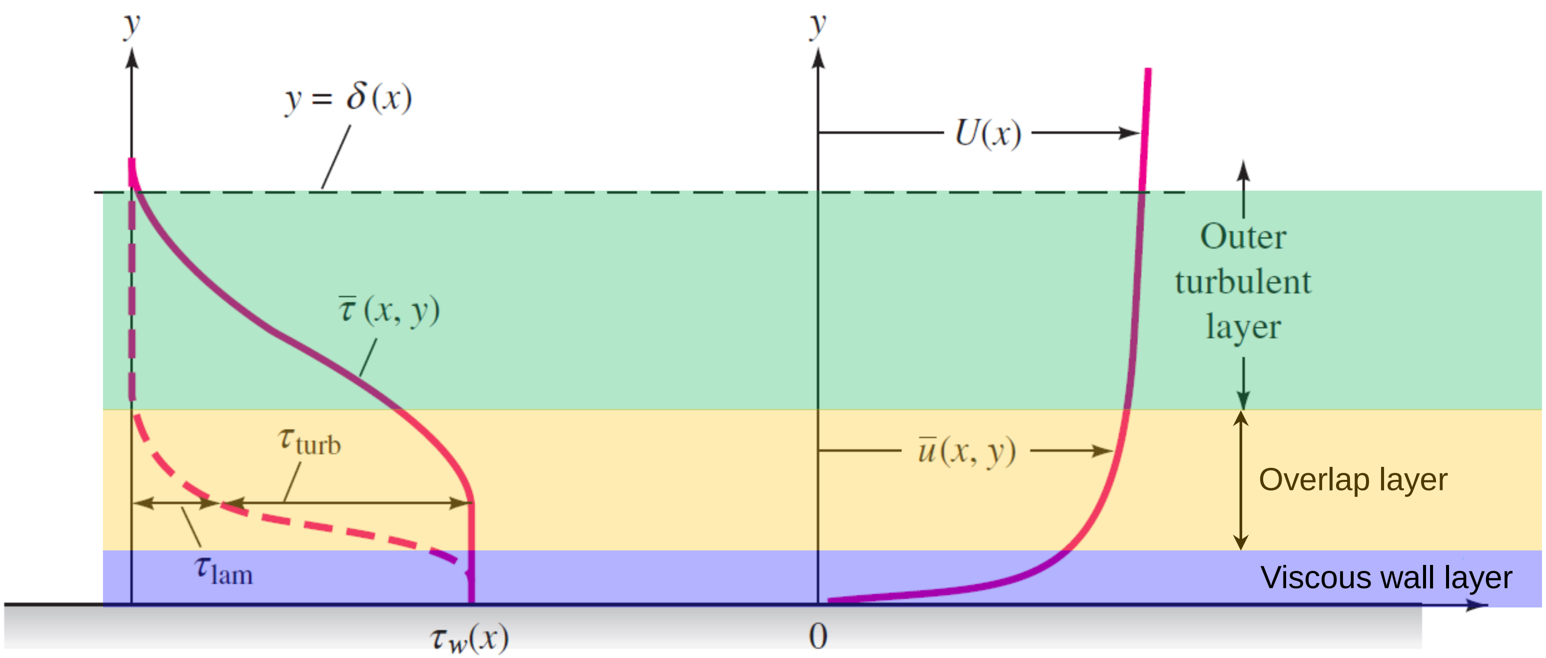


If we rewrite the Navier-Stokes equations replacing $\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}'$ and perform a time-average, we obtain that the shear stress in the boundary layer over a flat plate can be expressed as:

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} = \tau_{lam} + \tau_{turb}$$

Law of the wall

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'} = \tau_{lam} + \tau_{turb}$$

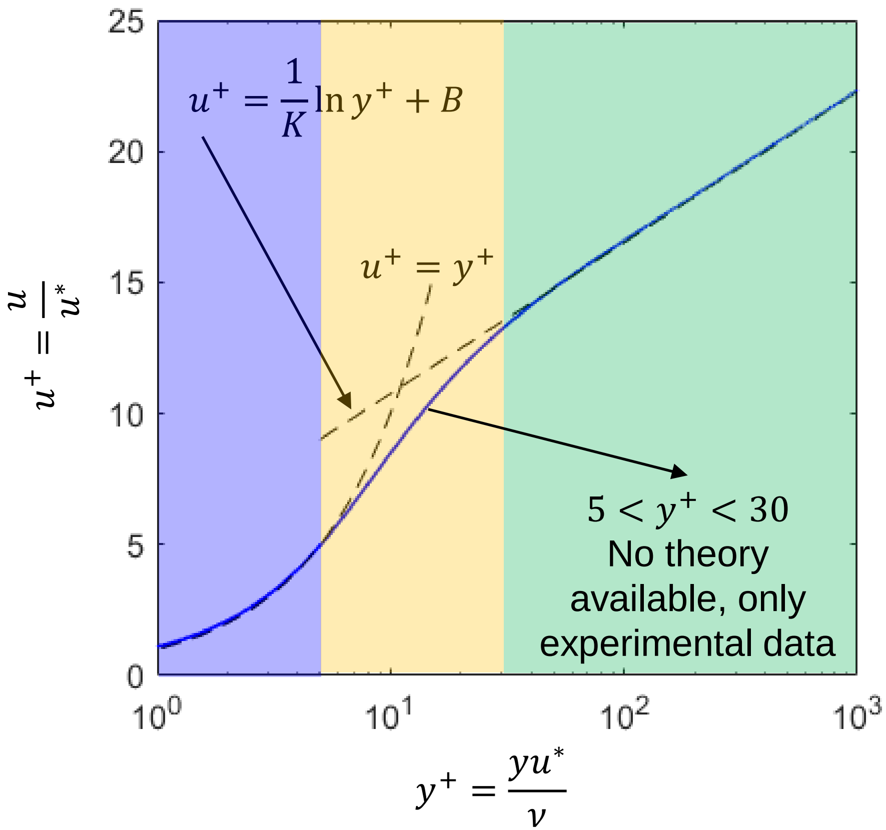


Based on the importance of τ_{lam} and τ_{turb}, the boundary layer in turbulent flow conditions can be decomposed into:

- Viscous sub-layer: near the wall, where laminar shear τ_{lam} dominates
- Overlap layer: both laminar and turbulent shear are important
- Turbulent layer: farther from the wall, where turbulent shear τ_{turb} dominates

Law of the wall

Experiments of turbulent wall flows over smooth walls, for both internal (pipes) and external flows, have demonstrated that there exists a universal velocity profile for the flow near the wall in many different flow configurations: the **law of the wall**



Non-dimensional distance from the wall:

$$y^+ = \frac{yu^*}{\nu}, \qquad u^* = \left(\frac{\tau_w}{\rho}\right)^{1/2}$$ Friction velocity

Non-dimensional velocity parallel to the wall: $u^+ = \frac{u}{u^*} = F(y^+)$

Law of the wall:

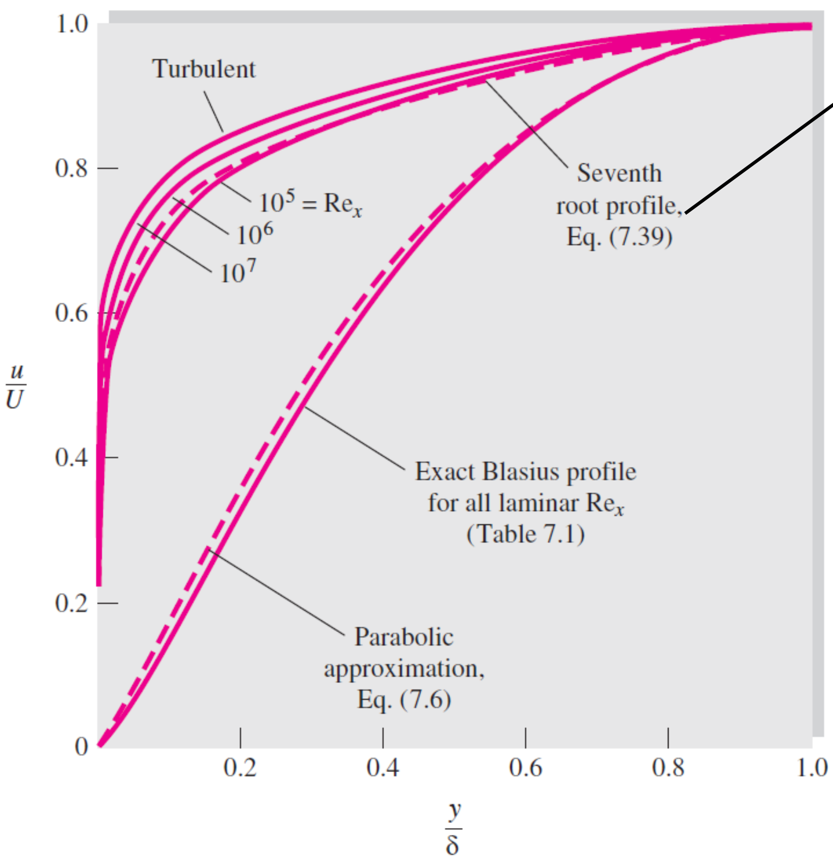
- $y^+ \leq 5$: viscous sublayer, $u^+ = y^+$
- $y^+ \geq 30$: turbulent layer, $u^+ = \frac{1}{K}\ln y^+ + B$,

from experiments (smooth wall): *K=0.4, B=5*

- $y^+ \gg 30$: the profile deviates from the log law and becomes case-dependent

Turbulent boundary layers - Prandtl

Prandtl compiled many experimental datasets for turbulent boundary layers over smooth plates and observed that a one-seventh-power-law approximates well the turbulent velocity profile (discarding the laminar sublayer):



$$\frac{u}{U_0} \approx \left(\frac{y}{\delta}\right)^{1/7}$$

Based on this, he obtained the following law for the film thickness (details in F. White, Sec. 7.4):

$$\frac{\delta}{x} \approx \frac{0.16}{Re_x^{1/7}}$$

and: $C_f(x) = \dfrac{0.027}{Re_x^{1/7}}, \qquad C_D = \dfrac{0.031}{Re_L^{1/7}}$

Summary of boundary layers over smooth walls

Summary of the correlations obtained for laminar and turbulent boundary layers over a smooth flat wall:

	Laminar, von Karman	Laminar, Blasius	Turbulent, Prandtl
Boundary layer thickness δ	$\frac{\delta}{x} = \frac{5.5}{\sqrt{Re_x}}$	$\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}}$	$\frac{\delta}{x} = \frac{0.16}{Re_x^{1/7}}$
Displacement thickness δ^*	$\frac{\delta^*}{x} = \frac{1.826}{\sqrt{Re_x}}$	$\frac{\delta^*}{x} = \frac{1.721}{\sqrt{Re_x}}$	$\frac{\delta^*}{x} = \frac{0.02}{Re_x^{1/7}}$
Momentum thickness θ	$\frac{\theta}{x} = \frac{0.73}{\sqrt{Re_x}}$	$\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$	$\frac{\theta}{x} = \frac{0.0156}{Re_x^{1/7}}$
Shape factor $H = \delta^*/\theta$	2.5	2.59	1.28
Skin friction coefficient C_f	$C_f = \frac{0.73}{\sqrt{Re_x}}$	$C_f = \frac{0.664}{\sqrt{Re_x}}$	$C_f = \frac{0.027}{Re_x^{1/7}}$
Drag coefficient C_D	$C_D = \frac{1.46}{\sqrt{Re_L}}$	$C_D = \frac{1.328}{\sqrt{Re_L}}$	$C_D = \frac{0.031}{Re_L^{1/7}}$

Worked example 7

From 2018/19 exam. A thin flat plate of 2.5 metre length is mounted parallel to the free stream in a wind tunnel, the free stream velocity is fixed to 42 m/s with high turbulence intensity. What would be the thickness of the boundary layer at the middle of the flat plate. (Take ρ $= 1.2\ kg/m^3$, $\mu = 1.8 \cdot 10^{-5} Pa \cdot s$).

Solution

"middle of the flat plate": $x = L/2$

$$Re_{x=L/2} = \frac{\rho U L/2}{\mu} = \frac{1.2\,\frac{kg}{m^3} \cdot 42\,\frac{m}{s} \cdot 1.25\,m}{0.000018\ Pa \cdot s} = 3.5 \cdot 10^6$$

→ Flow is turbulent

$$\delta(x = L/2) = \frac{0.16x}{Re_x^{1/7}} = 23.2\ mm$$

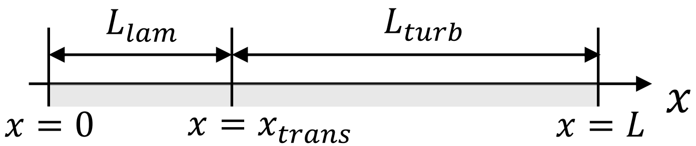
Drag coefficient for laminar-turbulent flows

There is no unique value for the laminar-to-turbulent flow transition over flat plates:

$$Re_{trans} = 5 \cdot 10^5 - 3 \cdot 10^6$$

Rough wall, disturbed flow ($5 \cdot 10^5$)

Smooth wall, well-organised flow ($3 \cdot 10^6$)



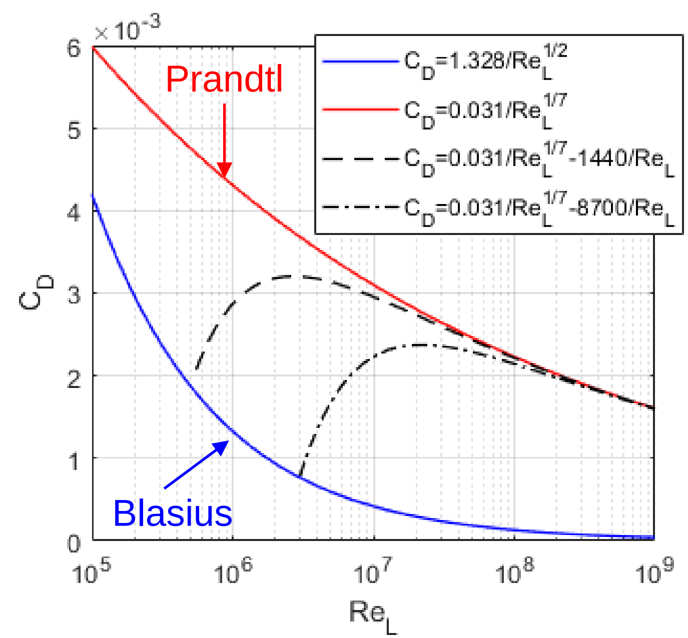
$$x_{trans} = \frac{\mu Re_{trans}}{\rho U}, \qquad L_{lam} = x_{trans}, \qquad L_{turb} = L - x_{trans}$$

If $Re_L < Re_{trans}$, it means that $L < x_{trans}$ and the plate is all in laminar flow; we can use (Blasius):

$$C_D = \frac{1.328}{\sqrt{Re_L}}$$

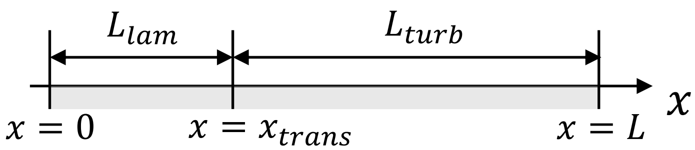
If $Re_L \gg Re_{trans}$, it means that $L_{turb} \approx L$; we can assume that the entire plate is in turbulent flow:

$$C_D = \frac{0.031}{Re_L^{1/7}}$$



Drag coefficient for laminar-turbulent flows

But, at intermediate values of Re_L ($Re_L > Re_{trans}$, but not too much) the extension of the section in laminar flow will not be negligible…Which formula for C_D should be used? We can think of an average value:

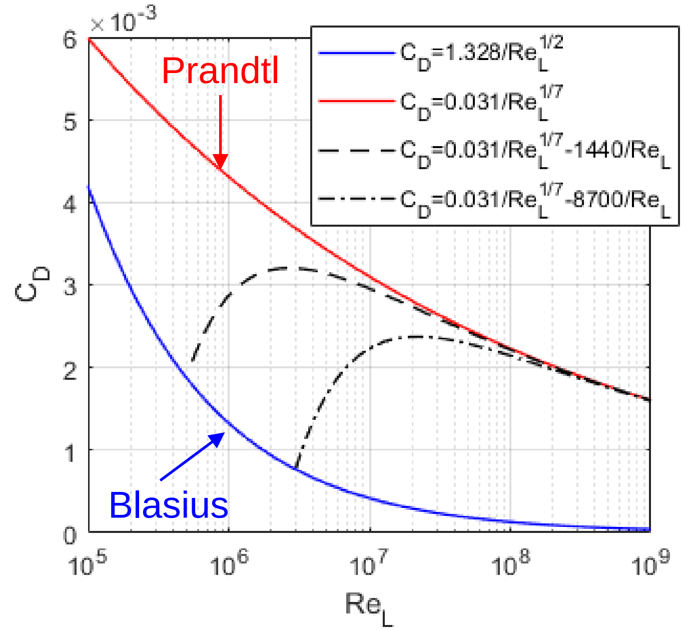


$$C_D = C_{D,lam}\frac{L_{lam}}{L} + C_{D,turb}\frac{L_{turb}}{L}$$

This has been already worked out by Schlichting, who proposed the following correlations:

$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{1440}{Re_L}, \qquad if\ Re_{trans} = 5 \cdot 10^5$$

$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{8700}{Re_L}, \qquad if\ Re_{trans} = 3 \cdot 10^6$$



As Re_L becomes very large, the two correlations converge to Prandtl formula (see figure).

Worked example 8

Take Worked example 7 and evaluate the drag force exerted on the plate assuming (a) turbulent flow from the leading edge, (b) laminar turbulent flow with $Re_{trans} = 5 \cdot 10^5$, (c) laminar turbulent flow with $Re_{trans} = 3 \cdot 10^6$. Consider a plate width of 0.5 m.

Solution

(a)

$$Re_L = \frac{\rho U L}{\mu} = \frac{1.2\,\frac{kg}{m^3} \cdot 42\,\frac{m}{s} \cdot 2.5\,m}{0.000018\,Pa \cdot s} = 7 \cdot 10^6$$

$$C_D = \frac{0.031}{Re_L^{1/7}} = 0.00326 \qquad D = \frac{1}{2} C_D \rho U_0^2 b L = 4.31\,N$$

(b)

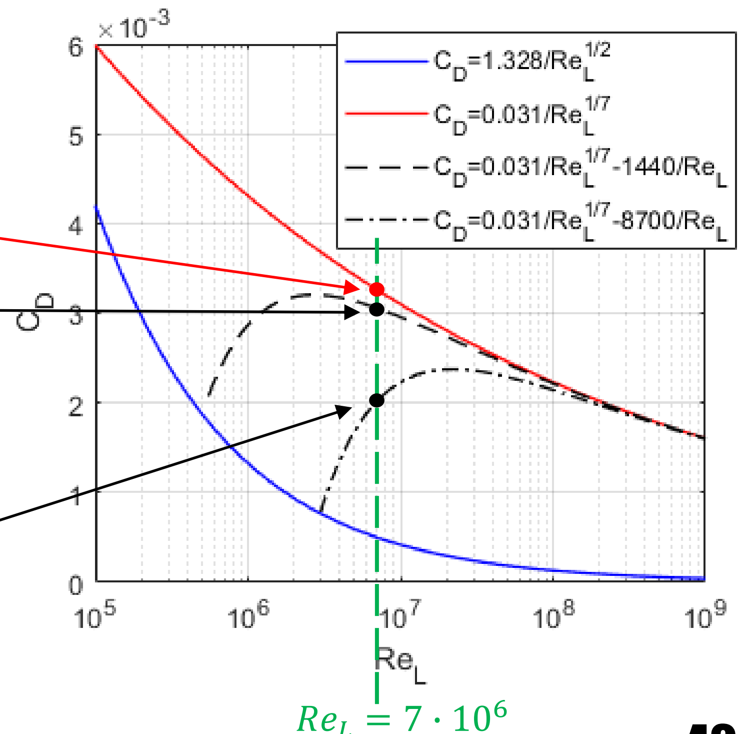
$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{1440}{Re_L} = 0.00306$$

$$D = 4.05\,N$$

(c)

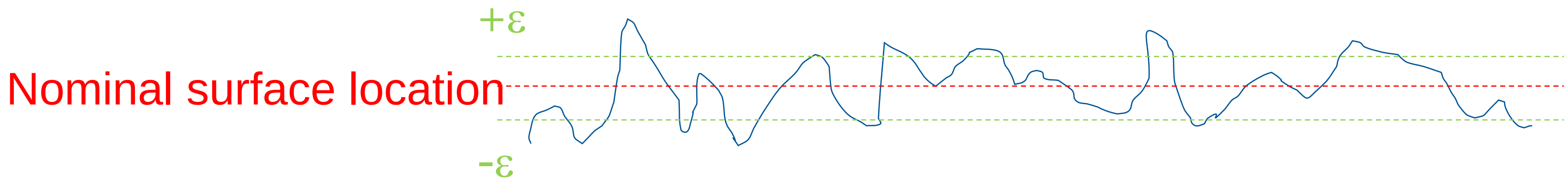
$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{8770}{Re_L} = 0.00201$$

$$D = 2.66\,N$$



Effect of roughness

Real surfaces are never perfectly smooth, but there is always a certain level of wall roughness ε, which depends on material, production process, operating conditions, etc.

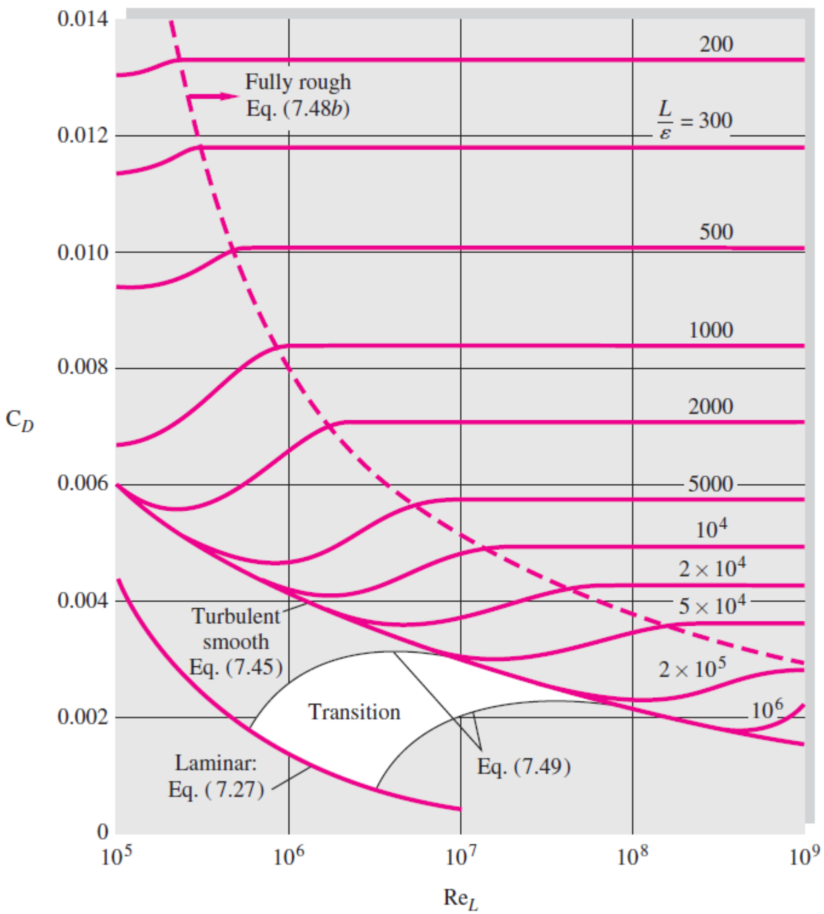


The wall roughness has no impact on the values of C_D when the flow is laminar. However, high roughness may cause early transition to turbulence, i.e. it reduces Re_{trans}. For turbulent flow, the surface roughness causes a deviation of C_D from the smooth wall correlation ($C_D = 0.031/Re_L^{1/7}$ for a flat plate), with C_D increasing with increasing ε/L.

Depending on the roughness parameter ε/L, C_D for flat plates in turbulent flow can be evaluated based on a chart which is the analogous of Moody chart for pipe flow (TF1 notes, p. 118).

Effect of roughness

Depending on the roughness parameter ε/L, C_D for flat plates in turbulent flow can be evaluated based on a chart which is the analogous of Moody chart for pipe flow (TF1 notes, p. 118).



Therefore, for rough plates in turbulent flow, $C_D = C_D(Re_L, \varepsilon/L)$. Furthermore, experiments have shown that, at fixed ε/L, there exists a threshold value of Re_L above which C_D becomes independent of Re_L, i.e. the curves in the chart become horizontal and $C_D = C_D(\varepsilon/L)$ only. Note that the same happens in pipe flows.

This regime is called **fully rough**, and in this regime C_D can be calculated as:

$$C_D = \left(1.89 + 1.62 \log \frac{L}{\varepsilon}\right)^{-2.5}$$

Worked example 9

Take Worked example 7. After years of use and poor maintenance, the flat plate (previously considered perfectly smooth) may have dirt deposited on it, which makes its surface appear as rough. Evaluate C_D assuming (a) $\varepsilon = 0.25\ mm$ and (b) $\varepsilon = 1.25\ mm$.

Solution

(a)

$$Re_L = \frac{\rho U L}{\mu} = 7 \cdot 10^6$$

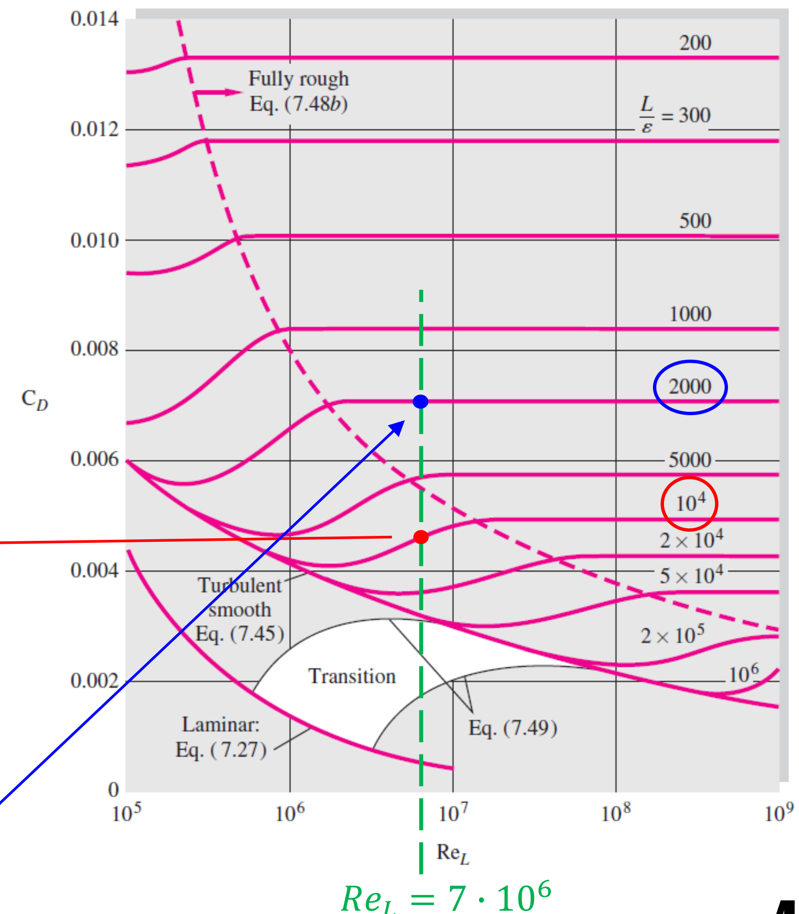
$$\frac{L}{\varepsilon} = \frac{2.5\ m}{0.00025\ m} = 10^4$$

$C_D = 0.0047$ (from chart)

(b)

$$\frac{L}{\varepsilon} = \frac{2.5\ m}{0.00125\ m} = 2000$$

$$C_D = \left(1.89 + 1.62 \log \frac{L}{\varepsilon}\right)^{-2.5} = 0.0071$$



Worked example 10

A hydrofoil 0.4 m long and 2 m wide is placed in a seawater flow of 12 m/s. Estimate the friction drag on both sides of the foil for (a) turbulent smooth-wall flow from the leading edge, (b) laminar turbulent flow with $Re_{trans} = 5 \cdot 10^5$, (c) turbulent rough-wall flow with $\varepsilon = 0.12\, mm$. Take: $\rho = 1025\ kg/m^3, \nu = 1.05 \cdot 10^{-6}\ m^2/s$

Solution

(a)

$$Re_L = \frac{UL}{\nu} = \frac{12\frac{m}{s} \cdot 0.4\, m}{1.05 \cdot 10^{-6}\ m^2/s} = 4.57 \cdot 10^6$$

$$C_D = \frac{0.031}{Re_L^{1/7}} = 0.00347 \qquad D = 2\left(\frac{1}{2} C_D \rho U_0^2 bL\right) = 410\ N$$

(b)

$$C_D = \frac{0.031}{Re_L^{1/7}} - \frac{1440}{Re_L} = 0.00315 \qquad D = 2\left(\frac{1}{2} C_D \rho U_0^2 bL\right) = 372\ N$$

(c)

$$\frac{L}{\varepsilon} = \frac{0.4\, m}{0.00012\, m} = 3333$$ → Fully rough regime

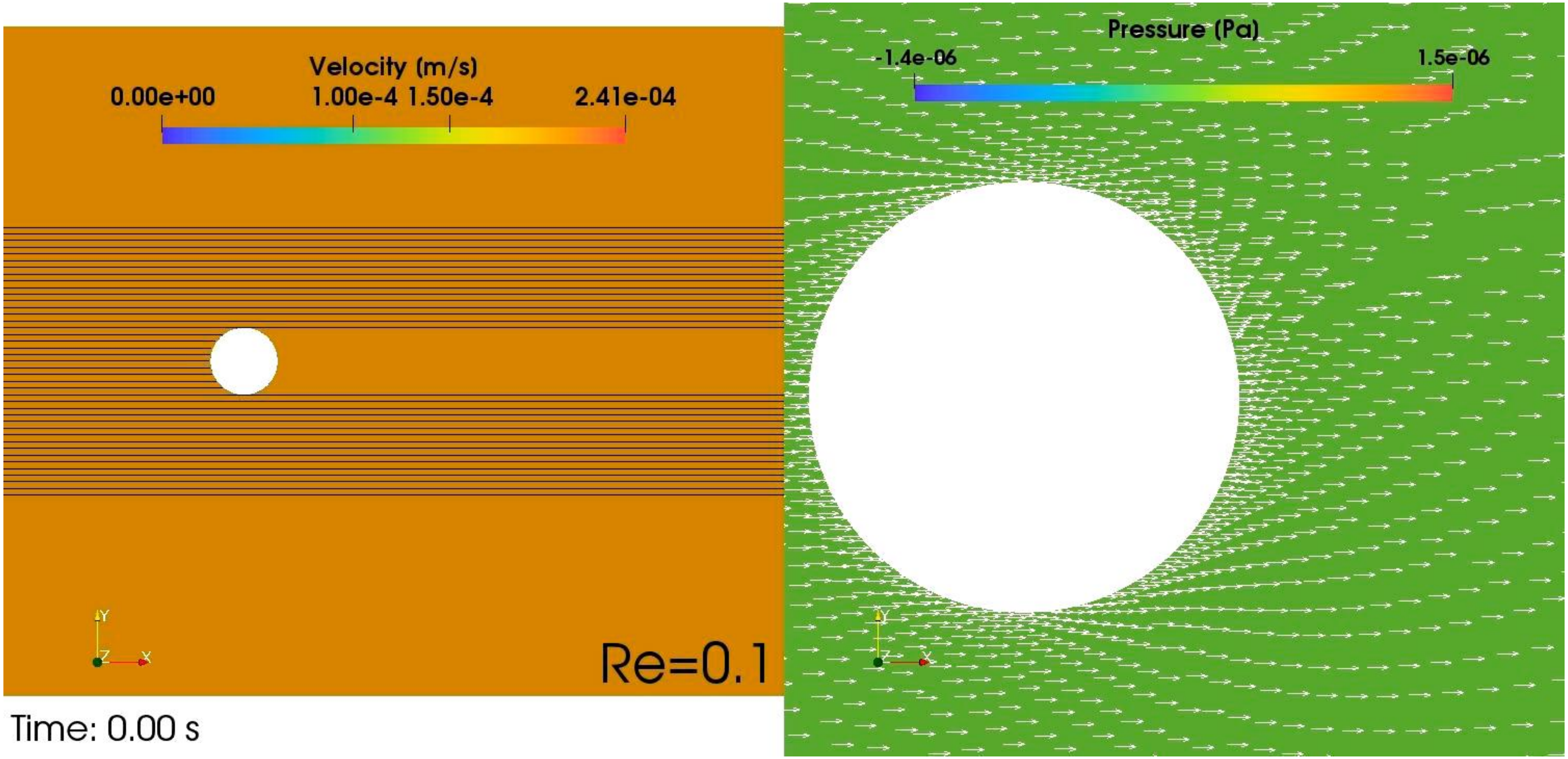
$$C_D = \left(1.89 + 1.62 \log \frac{L}{\varepsilon}\right)^{-2.5} = 0.00629 \qquad D = 2\left(\frac{1}{2} C_D \rho U_0^2 bL\right) = 743\ N$$

Flow past cylinder

https://www.dropbox.com/s/hd8toebmkg5i2zq/flowPastCylinder_AllRe.avi?dl=0

$$Re_D = \frac{\rho U D}{\mu}$$

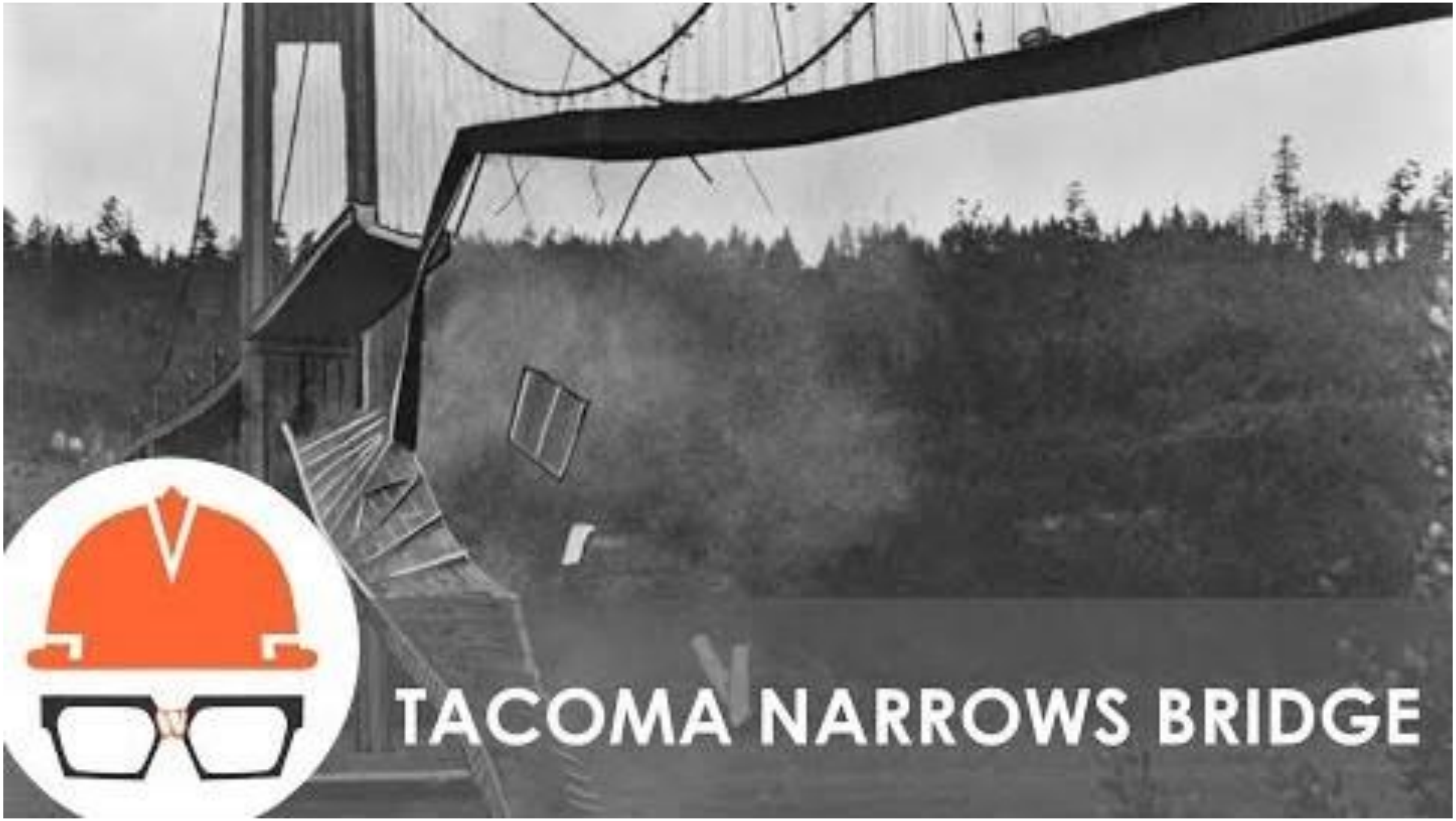
$Re_D \geq 80$: von Karman vortex street. Flow is still laminar!



Time: 0.00 s

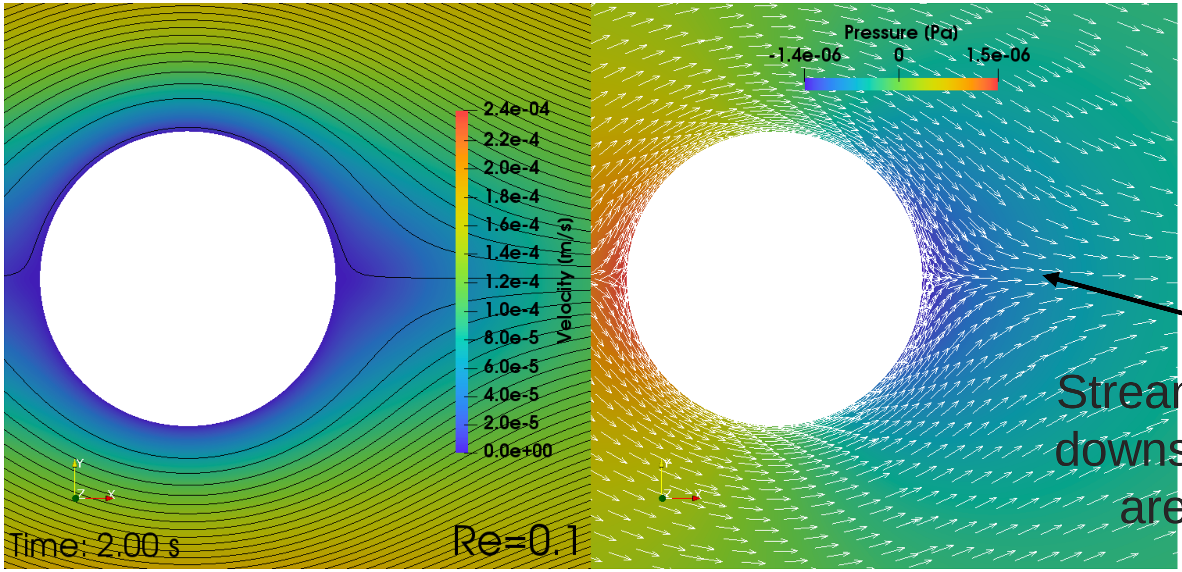
Flow past cylinder

Do you think that von Karman vortex street is simply nice colors? Take a look at the Tacoma bridge collapse: https://www.youtube.com/watch?v=mXTSnZgrfxM



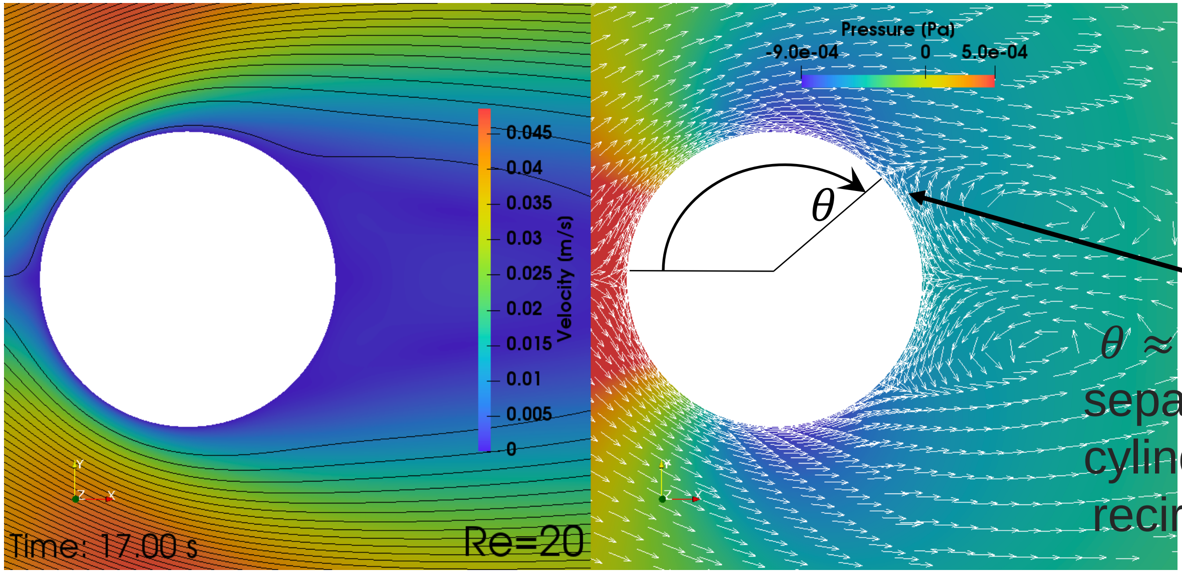
Flow past cylinder

$Re_D = 0.1$



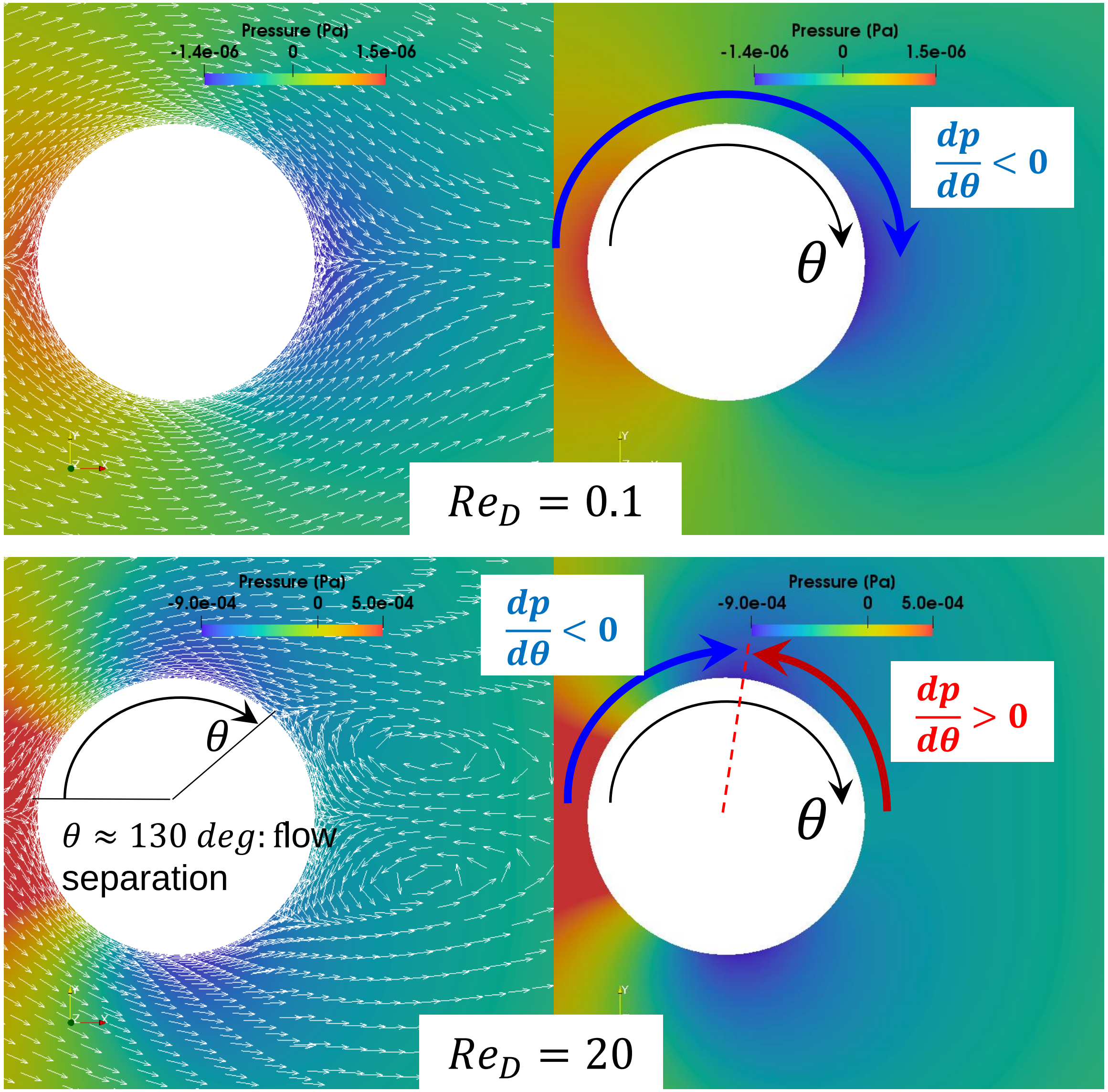
Streamlines and vectors downstream the cylinder are well organised

$Re_D = 20$



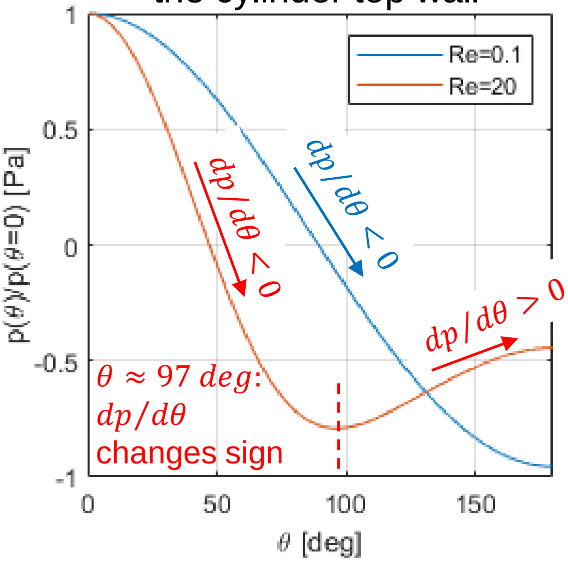
$\theta \approx 130\ deg$: flow separates from the cylinder, creating a recirculating wake

Flow past cylinder – effect of pressure gradient



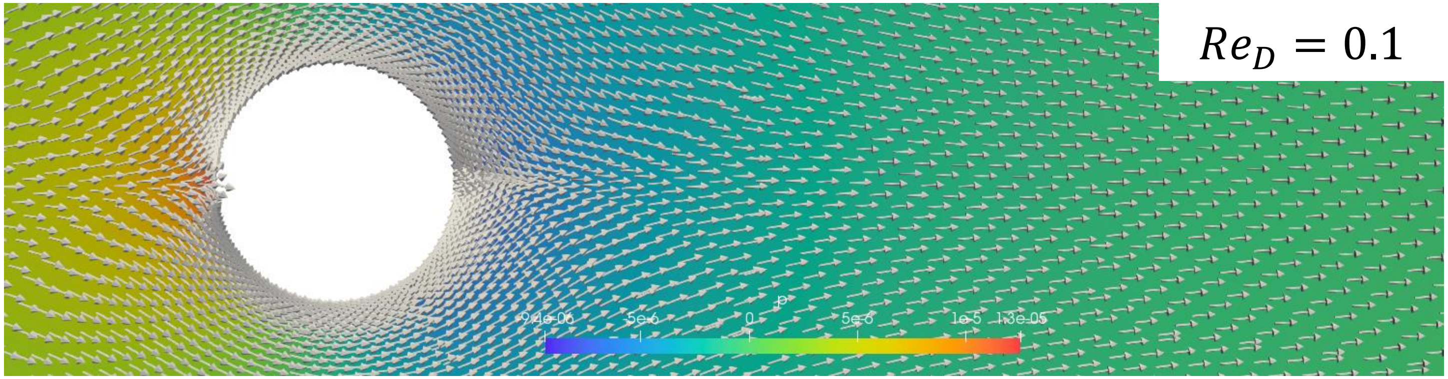
$\frac{dp}{d\theta} < 0$: Favourable pressure gradient that promotes boundary layer attachment

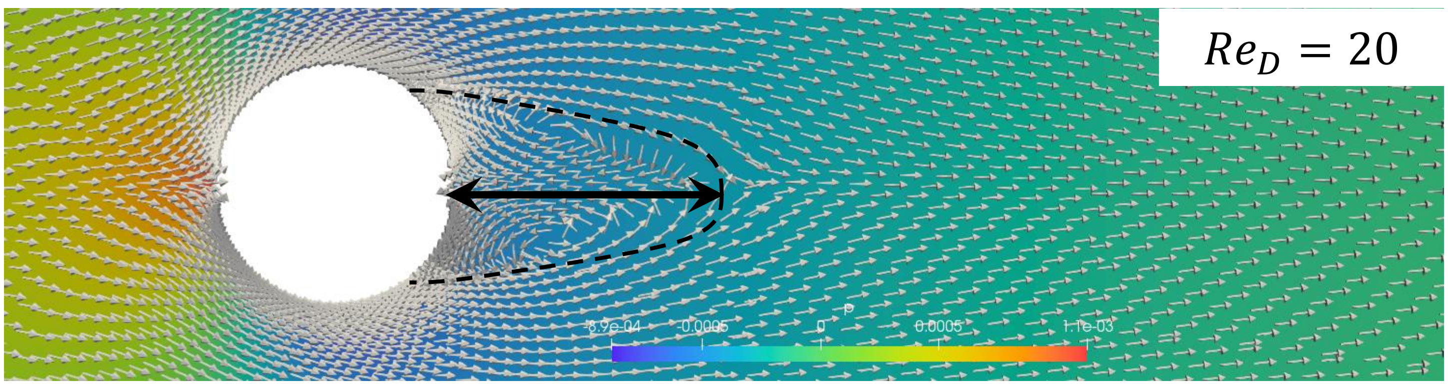
Pressure profile along the cylinder top wall

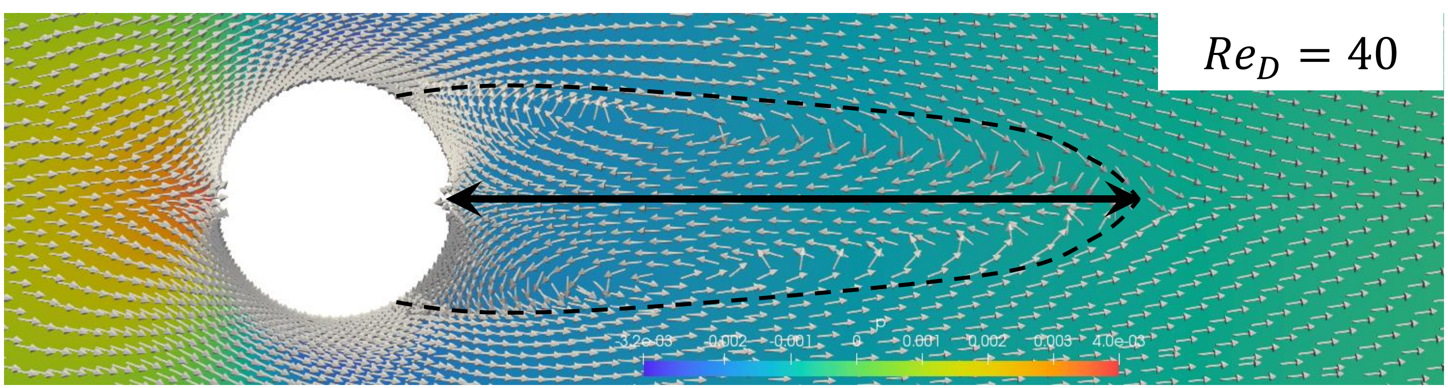


$\frac{dp}{d\theta} > 0$: Adverse pressure gradient that induces flow separation

Flow past cylinder – wake length

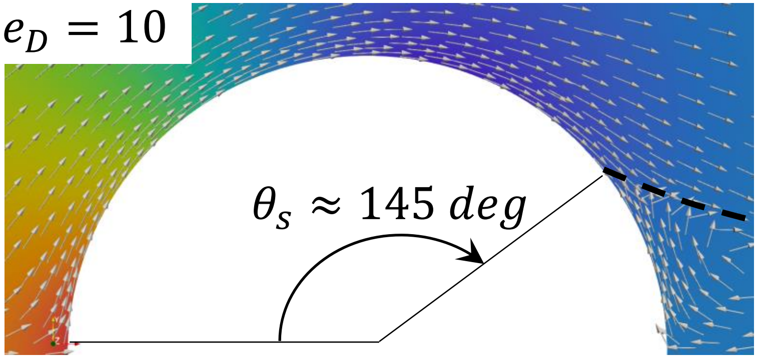




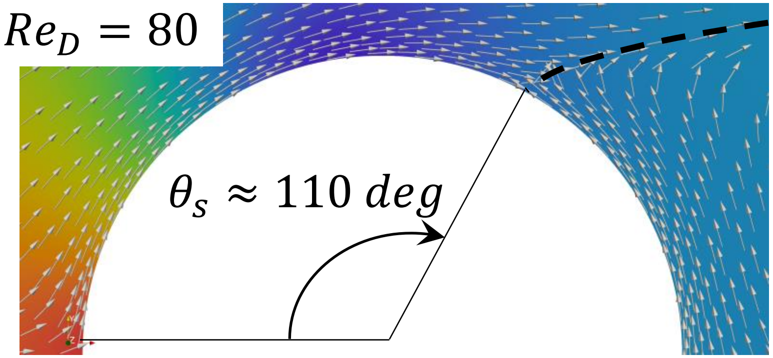


Flow past cylinder – flow separation

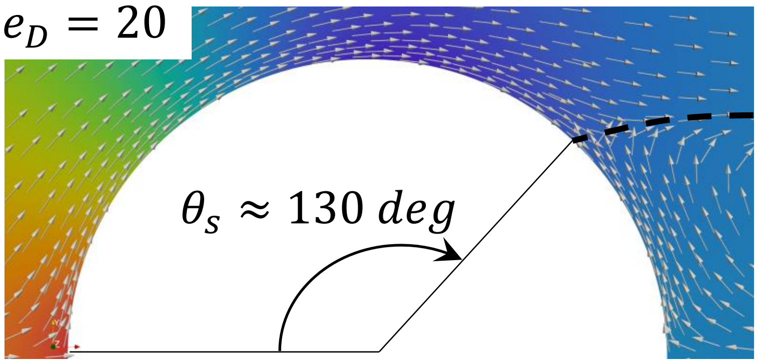
$Re_D = 10$



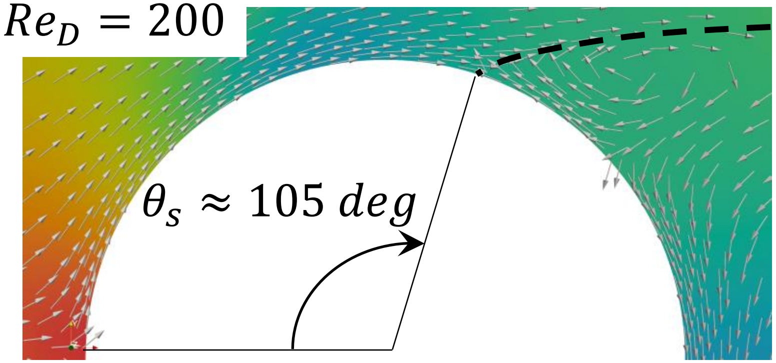
$Re_D = 80$



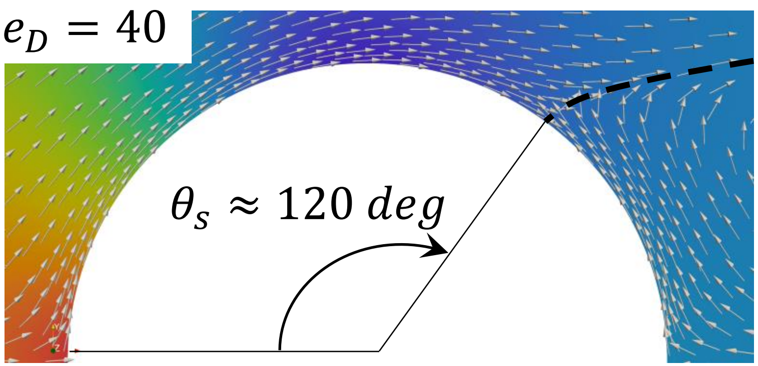
$Re_D = 20$

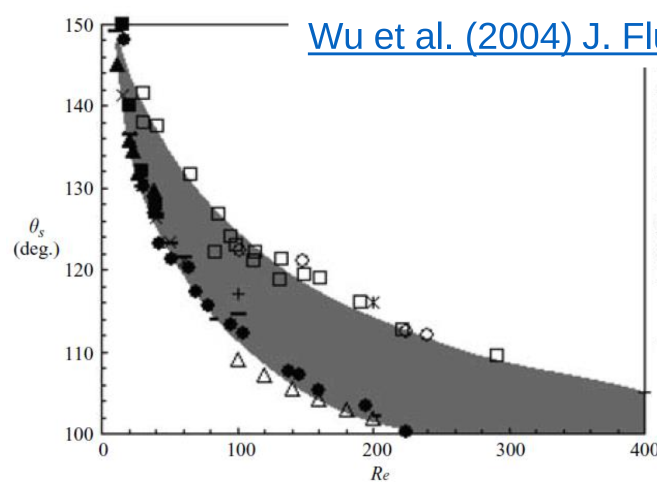


$Re_D = 200$



$Re_D = 40$





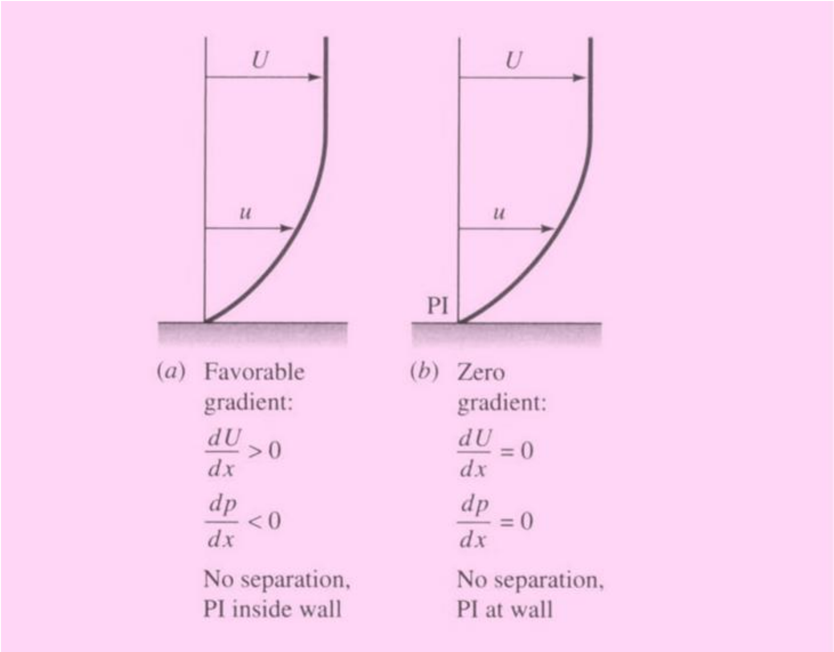
Wu et al. (2004) J. Fluid Mech. 515:233

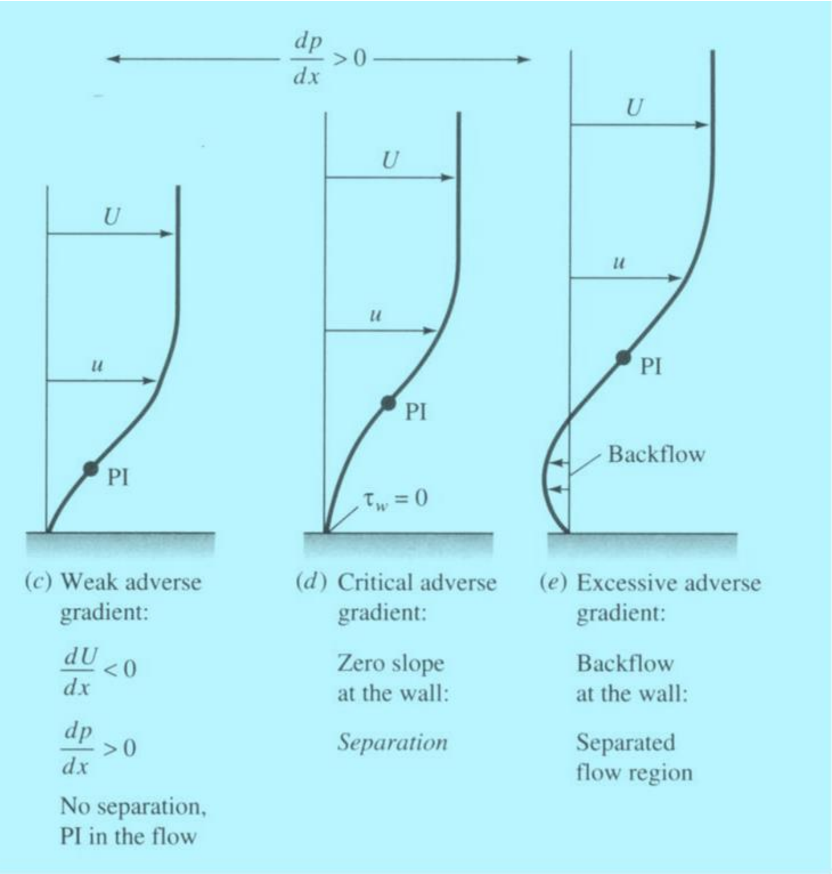
θ_s decreases as Re_D increases

Effect of pressure gradient

Free-stream velocity and pressure gradient are related by the Bernoulli equation:

$$p + \frac{\rho U^2}{2} = const \Rightarrow \frac{dp}{dx} + \rho U \frac{dU}{dx} = 0 \Rightarrow \boxed{\frac{dp}{dx} = -\rho U \frac{dU}{dx}}$$





Separation occurs when the adverse pressure gradient is too large for the fluid to continue flowing along the wall. In general, separation depends on Re, the shape of the body and the relative direction of the flow compared to the surface of the body

Topic 2 – Boundary layer flows

Now you should be able to:

- Sketch laminar and turbulent boundary layers over flat plates
- Explain the difference among boundary layer thickness, displacement & momentum thickness, and calculate them based on laminar/turbulent flow correlations
- Explain the relationship between momentum thickness, sheat stress, drag force and drag coefficient for a flat plate, and calculate the drag force for laminar/turbulent flow
- Sketch the velocity profile in a turbulent boundary layer based on the law of the wall
- Understand how roughness affects the drag force
- Describe how the pressure gradient impacts the boundary layer profile

Further reading/assessment:

- F. White book, Sec. 7.1, 7.2, 7.3, 7.4, 7.5, 6.5 (law of the wall); problems in Ch. 7.

Seminar

Worked example 11

A train is 100 m long, 2.8 m wide and 2.75 m high. The train travels at 180 km/h through air of density 1.2 kg/m^3 and kinematic viscosity 1.5·10^{-5} m^2/s. Calculate (a) the boundary layer thickness at the rear of the train, (b) the frictional drag acting on the train, (c) the power required to overcome the frictional drag.

Solution

Worked example 12

Exam 15/16: fluids longer question 20.

(a) A laminar flow of air moves parallel to a flat plate of length $L = 0.6\ m$ and width $b = 0.3\ m$, with a velocity $U = 5\ m/s$. Estimate at the trailing edge of the plate, the boundary layer thickness, δ, the displacement thickness, δ^*, and the momentum thickness, θ. In your estimation use Blasius boundary layer exact solution. For air, take $\rho = 1.2\ kg/m^3$ and $\mu = 1.8 \cdot 10^{-5}\ Pa \cdot s$.

Solution – We did this already, worked example 3

Worked example 12

Exam 15/16: fluids longer question 20.

(b) If the flow field above the flat plate increases its velocity and becomes turbulent, you can use a strain gauge to measure the surface shear stress, τ_w, at a position x from the leading edge of the plate, in order to estimate the new value of the flow velocity U above the turbulent boundary layer. By considering Prandtl's boundary layer approximation, show that in this case the following relation can be obtained between the surface shear, τ_w, and external flow velocity, U:

$$\tau_w = \frac{0.0134\rho^{6/7}\mu^{1/7}}{x^{1/7}} U^{13/7}$$

Formulae sheet from Moodle

Solution

Worked example 12

Exam 15/16: fluids longer question 20.

(c) How much is the new value of U, if in your measurement you found that at the trailing edge of the plate, $x = 0.6\ m$, the value of $\tau_w = 20\ kg/(ms^2)$. Also for this new value of U, estimate at the trailing edge of the plate, the boundary layer thickness, δ, the momentum thickness, θ, and the total drag force D supported by the plate. In your estimation use Prandtl's turbulent boundary layer approximation.

Solution

Worked example 12

Exam 15/16: fluids longer question 20.

(d) Finally find the corresponding turbulent boundary layer flow velocity "u" at the trailing edge of the plate $x = 0.6\,m$, and at the height y given by your previous estimation in part a) of the laminar boundary layer thickness. In your evaluation of "u" use the power law approximation of turbulent boundary layer velocity profile:

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$$

Solution

Worked example 13

Consider the following expression for the velocity distribution in a laminar boundary layer on a flat plate:

$$\frac{u(x,y)}{U_0} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3, \qquad 0 \le y \le \delta(x)$$

(a) Is this expression satisfying the required boundary conditions for the velocity field?

(b) Derive relationships for the boundary layer thickness, displacement thickness, momentum thickness and drag coefficient based on this velocity profile.

Solution

Worked example 13

Worked example 13